ON THE APPROXIMATION OF A FUNCTION CONTINUOUS OFF A CLOSED SET BY ONE CONTINUOUS OFF A POLYHEDRON

STEVEN P. ELLIS

Abstract. Let P be a finite simplicial complex (i.e., a finite collection of simplices that fit together nicely) and denote its underlying space (the union of the simplices in P) by $|P|$. Let Q be a subcomplex of P (e.g., $Q = P$). Let $a \geq 0$. Then there exists $K < \infty$, *depending only on a and Q*, with the following property. Let $\mathcal{S} \subset |P|$ be closed and suppose Φ is a continuous map of $|P| \setminus \mathcal{S}$ into some topological space F. ("$\setminus$" indicates set-theoretic subtraction.) Suppose $\dim(\tilde{\mathcal{S}} \cap |Q|) \leq a$, where "dim" indicates Hausdorff dimension. Then there exists $\tilde{\mathcal{S}} \subset |P|$ such that $\tilde{\mathcal{S}} \cap |Q|$ is the underlying space of a subcomplex of Q and there is a continuous map $\tilde{\Phi}$ of $|P| \setminus \tilde{\mathcal{S}}$ into F such that

- $\mathcal{H}^a\big(\tilde{\mathcal{S}} \cap |Q|\big) \leq K \mathcal{H}^a\big(\mathcal{S} \cap |Q|\big)$, where $\mathcal{H}^a$ denotes a-dimensional Hausdorff measure;
- if $x \in \tilde{\mathcal{S}}$ then x belongs to a simplex in P intersecting $\mathcal{S}$;
- if $x \in |P| \setminus \mathcal{S}$, $x \in \sigma \in P$, and σ does not intersect any simplex in Q whose simplicial interior intersects $\mathcal{S}$, then $\tilde{\Phi}(x)$ is defined and equals $\Phi(x)$;
- if $\sigma \in P$ then $\tilde{\Phi}(\sigma \setminus \tilde{\mathcal{S}}) \subset \Phi(\sigma \setminus \mathcal{S})$;
- if F is a metric space and Φ is locally Lipschitz on $|P| \setminus \mathcal{S}$ then $\tilde{\Phi}$ is locally Lipschitz on $|P| \setminus \tilde{\mathcal{S}}$; and
- $\dim(\tilde{\mathcal{S}} \cap |Q|) \leq \dim(\mathcal{S} \cap |Q|)$ and $\dim \tilde{\mathcal{S}} \leq \dim \mathcal{S}$.

Moreover, P can be replaced by an arbitrarily fine subdivision without changing the constant K. Consequently, modulo subdivision, if $\epsilon > 0$, we may assume $\tilde{\Phi}(x) = \Phi(x)$ if $dist(x, \mathcal{S})$, the distance from x to $\mathcal{S}$, exceeds ϵ and we may assume $\max\big\{dist(y, \mathcal{S}) : y \in \tilde{\mathcal{S}}\big\} < \epsilon$.

Note that $\mathcal{S}$ can be any compact subset of $|P|$. For example, no rectifiability assumptions on $\mathcal{S}$ are required. But $\tilde{\mathcal{S}}$ is rectifiable and if M is a compact C^1 manifold, $\mathcal{T} \subset M$ is closed, and $\Phi : M \setminus \mathcal{T} \to \mathsf{F}$ is continuous, then it is immediate from the preceding that Φ can be approximated by a continuous map $\tilde{\Phi} : M \setminus \tilde{\mathcal{T}} \to \mathsf{F}$ where $\tilde{\mathcal{T}}$ is closed and a-rectifiable.

1. Introduction and main result

Let P be a finite simplicial complex of dimension p and denote its underlying space by $|P|$. (See appendix B for definitions and basic properties related to simplices and simplicial complexes.) Suppose $\mathcal{S} \subset |P|$ is compact and Φ maps $|P| \setminus \mathcal{S}$ continuously into a topological space F. ("$\setminus$" denotes set-theoretic subtraction) One might be interested in properties of $\mathcal{S}$. (E.g., see remark 1.7 below.) However, being a *prima facie* arbitrary compact subset of $|P|$, $\mathcal{S}$ may be hard to analyze. In this paper, we study the problem of deforming $\mathcal{S}$ and Φ a little so that $\mathcal{S}$ becomes a rather regular set, $\tilde{\mathcal{S}}$, specifically, the underlying space of a finite simplicial complex. By deforming "a little" I mean that the region of $|P|$ in which the deformation takes

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place can be confined to an arbitrary open neighborhood of $\mathcal{S}$ and the volume of $\tilde{\mathcal{S}}$ is controlled as well. ([Ell11] is a less detailed version of this paper.)

The idea of deforming a set into a union of cells is reminiscent of the "Deformation Theorem" in geometric measure theory (Federer [Fed69, pp. 401 – 408], Simon [Sim83, 29.1, p. 163 and 29.4, p. 166], Hardt and Simon [HS86, Hardt's Lecture 3, pp. 83–93] and Giaquinta *et al* [GMS98, Lemma 2, p. 495; Theorem 1, p. 498; and Theorem 2, p. 503, Volume I]). In our case it is not just $\mathcal{S}$ that gets deformed. Φ does as well. As explained in remark 1.11 below, it seems that our main theorem, theorem 1.1, does not easily follow from the Deformation Theorem nor will the method of proof of the Deformation Theorem work for the problem considered here.

If $a \geq 0$, let $\mathcal{H}^a$ denote a-dimensional Hausdorff measure. (See subsection C in the appendix for background on Hausdorff measure and dimension.) Let $\lfloor a \rfloor$ denote the integer part of a, i.e., $\lfloor a \rfloor$ is the largest integer $\leq a$. The (nonconstructive) proof of the following can be found in section 2 (with the most technical aspects of the proof relegated to appendix A). If X is a metric space with metric d, $x \in X$, and $A \subset X$, let

$$dist(x, A) = \inf\{d(x, y) : y \in A\}.$$

Also recall that the "diameter" of A is defined to be

$$diam(A) = \sup\{d(x, y) \geq 0 : x, y \in A\}.$$

(See Munroe [Mun71, p. 12] .) Given A, the function $x \mapsto dist(x, A)$ is obviously continuous, in fact, Lipschitz (appendix C), in $x \in X$. (See Munroe [Mun71, p. 12].)

In the following "dim" denotes Hausdorff dimension and, for $a \geq 0$, $\mathcal{H}^a$ denotes a-dimensional Hausdorff measure. (See appendix C.)

Theorem 1.1. *Let P be a finite simplicial complex lying in a Euclidean space. Let $|P|$ be the polytope or underlying space of P. Use the metric on $|P|$ that it inherits from the ambient Euclidean space. Let $\mathcal{S} \subset |P|$ be closed. Let F be a topological space and suppose $\Phi : |P| \setminus \mathcal{S} \to \mathsf{F}$ is continuous. Let Q be a subcomplex of P (e.g., $Q = P$), let $a \geq 0$, and suppose* $\dim(\mathcal{S} \cap |Q|) \leq a$. *Then there is a closed set, $\tilde{\mathcal{S}} \subset |P|$ and a continuous map $\tilde{\Phi} : |P| \setminus \tilde{\mathcal{S}} \to \mathsf{F}$ such that*

(1) *If F is a metric space and Φ is locally Lipschitz off $\mathcal{S}$ then $\tilde{\Phi}$ is locally Lipschitz off $\tilde{\mathcal{S}}$.*
(2) $\dim(\tilde{\mathcal{S}} \cap |Q|) \leq \dim(\mathcal{S} \cap |Q|)$ *and* $\dim \tilde{\mathcal{S}} \leq \dim \mathcal{S}$.
(3) *$\tilde{\mathcal{S}} \cap |Q|$ is either empty or the underlying space of a subcomplex of the $\lfloor a \rfloor$-skeleton of Q.*
(4) *Suppose $\tau \in P$ has the following property. If $\rho \in Q$ and $(Int\,\rho) \cap \mathcal{S} \neq \varnothing$ then $\tau \cap \rho = \varnothing$. Then $\tilde{\mathcal{S}} \cap \tau = \mathcal{S} \cap \tau$ and $\tilde{\Phi}$ and Φ agree on $\tau \setminus \mathcal{S}$.*
(5) *Let $\rho \in P \setminus Q$. (But $\rho \cap |Q| \neq \varnothing$ is possible.) Then for every $s \geq 0$, if $\mathcal{H}^s(\mathcal{S} \cap (Int\,\rho)) = 0$ then $\mathcal{H}^s(\tilde{\mathcal{S}} \cap (Int\,\rho)) = 0$. In particular,* $\dim(\tilde{\mathcal{S}} \cap (Int\,\rho)) \leq \dim(\mathcal{S} \cap (Int\,\rho))$.
(6) *If $\tau \in Q$ and $\mathcal{H}^{\lfloor a \rfloor}(\tilde{\mathcal{S}} \cap (Int\,\tau)) > 0$, then τ is an $\lfloor a \rfloor$-simplex and $\mathcal{H}^{\lfloor a \rfloor}[\mathcal{S} \cap (Int\,\sigma)] > 0$ for some simplex σ of Q having τ as a face. ($\sigma = \tau$ is possible.)*
(7) *If $y \in \tilde{\mathcal{S}}$ then there exists $\sigma \in P$ such that $y \in \sigma$ and $\sigma \cap \mathcal{S} \neq \varnothing$. Thus, $dist(y, \mathcal{S})$ is not greater than the largest of the diameters of the simplices in P.*
(8) *If $\sigma \in P$ then $\tilde{\Phi}(\sigma \setminus \tilde{\mathcal{S}}) \subset \Phi(\sigma \setminus \mathcal{S})$.*
(9) *There is a constant $K < \infty$ depending only on a and Q such that*

$$\mathcal{H}^a(\tilde{\mathcal{S}} \cap |Q|) \leq K \mathcal{H}^a(\mathcal{S} \cap |Q|). \tag{1.1}$$

(10) *There is a constant $K < \infty$ depending only on a and P with the following property. For every $\epsilon > 0$ there is a subdivision, P', of P such that diam$(\zeta) < \epsilon$ for every $\zeta \in P'$ and parts (1) through (8) above and* (1.1) *hold when P is replaced by P' and Q is replaced by the corresponding subcomplex of P' (subdivision of Q).*

The assumption that P is finite can be replaced by a regularity property trivially satisfied by finite complexes. The proof of the following is given in appendix A. (See appendix B for discussion of the topology of polytopes.)

Corollary 1.2. *Let P be a, not necessarily finite, simplicial complex s.t. $|P| \subset \mathbb{R}^N$, where $N < \infty$. Suppose*

(1.2) *Every $x \in |P|$ has a neighborhood, open in $\mathbb{R}^N$, intersecting only finitely many simplices in P.*

Then $|P|$ is a locally compact subspace of $\mathbb{R}^N$. (I.e., $|P|$'s polytope topology and relative topologies coincide and are locally compact.) Put on $|P|$ the restriction of the usual metric on $\mathbb{R}^N$. Let $\mathcal{S} \subset |P|$ be closed. Let F be a topological space and suppose $\Phi : |P| \setminus \mathcal{S} \to \mathsf{F}$ is continuous. Let Q be a finite subcomplex of P, let $a \geq 0$, and suppose $\dim\big(\mathcal{S} \cap |Q|\big) \leq a$. *Then there is a closed set, $\tilde{\mathcal{S}} \subset |P|$ and a continuous map $\tilde{\Phi} : |P| \setminus \tilde{\mathcal{S}} \to \mathsf{F}$ such that parts (1) through (9) of theorem 1.1 hold.*

Let P_Q be the simplicial complex consisting of all simplices in P that intersect $|Q|$ and the faces of all such simplices. Then P_Q is finite. The following replacement for part (10) of the theorem holds.

(10') *There is a constant $K < \infty$ depending only on a and P_Q with the following property. For every $\epsilon > 0$ there is a subdivision, P', of P such that diam$(\zeta) < \epsilon$ for every $\zeta \in P'$ with $\zeta \subset |P_Q|$ and parts (1) through (8) of theorem 1.1 and* (1.1) *hold when P is replaced by P' and Q is replaced by the corresponding subcomplex of P' (subdivision of Q).*

Remark 1.3. Suppose $a \geq 0$ is an integer. The theorem tells us that we may assume that Φ is continuous off an $\mathcal{H}^a$-measurable countably a-rectifiable set (Giaquinta *et al* [GMS98, pp. 90–91, Volume I], Hardt and Simon [HS86, p. 20]), and still have some control over its volume. In fact, by (C.7) in appendix C, trivially the same thing holds if $|P|$ is replaced by any compact space with a bi-Lipschitz triangulation (i.e., a triangulation that is Lipschitz and has a Lipschitz inverse), e.g., a compact C^1 manifold (Munkres [Mun66, Theorem 10.6, pp. 103–104]). (By Munkres [Mun84, Lemma 2.5, p. 10], any simplicial complex P with $|P|$ compact must be finite.)

Remark 1.4. We make the following simple observations.

1. From appendix C, we see that if we rescale P by multiplying by a constant $\lambda > 0$, then the constant K in (1.1) is multiplied by λ^a.
2. Part (8) of the theorem does not imply that $\tilde{\Phi}(x)$ and $\Phi(x)$ are close (when defined) because if $\sigma \in P$, $\Phi(\sigma \setminus \mathcal{S})$ may be big.
3. Suppose a is an integer. If $\mathcal{H}^a\big(\mathcal{S} \cap |Q|\big)$ is very small we must have $\dim(\tilde{\mathcal{S}} \cap |Q|\big) < a$. For suppose $\dim(\tilde{\mathcal{S}} \cap |Q|\big) = a$ then from theorem 1.1 parts (3 and 9), one can conclude

$$\mathcal{H}^a\big(\mathcal{S} \cap |Q|\big) \geq K^{-1}\mathcal{H}^a\big(\tilde{\mathcal{S}} \cap |Q|\big) \geq K^{-1}\mathcal{H}^a(\text{smallest } a\text{-simplex in } Q) > 0. \tag{1.3}$$

(4) Let P be a finite simplicial complex and let $\mathcal{S} \subset |P|$ be compact and have empty interior (in particular $\mathcal{S} \neq |P|$). One can easily construct a continuous function $\Phi : |P| \setminus \mathcal{S} \to \mathbb{R}$ such that Φ cannot be continuously extended to any set larger than $|P| \setminus \mathcal{S}$. Just take $\mathsf{F} = \mathbb{R}$ and

$$\Phi(x) = \sin\left(\frac{1}{dist(x, \mathcal{S})}\right), \quad x \in P \setminus \mathcal{S}.$$

(*Proof:* It is easy to see that, since $|P|$ is locally arcwise connected (appendix B), arbitrarily close to $x \in \mathcal{S}$ there are points $y, y' \in |P| \setminus \mathcal{S}$ such that (s.t.) $\sin\bigl[1/dist(y, \mathcal{S})\bigr] = +1$ and $\sin\bigl[1/dist(y', \mathcal{S})\bigr] = -1$.)

A potentially useful corollary is the following. See appendix A for the proof.

Corollary 1.5. *Let P be a finite simplicial complex and let $a \geq 0$. Then there exists $K < \infty$ (depending only on a and P) with the following property. Let $\mathcal{S} \subset |P|$ be compact and suppose Φ is a continuous map of $|P| \setminus \mathcal{S}$ into some topological space* F. *Suppose* $\dim \mathcal{S} \leq a$. *Let $\epsilon > 0$. Then there exists $\tilde{\mathcal{S}} \subset |P|$ and there is a continuous map $\tilde{\Phi}$ of $|P| \setminus \tilde{\mathcal{S}}$ into* F *such that*

(1) *$\tilde{\mathcal{S}}$ is the underlying space of a subcomplex of a subdivision of P.*
(2) $\max\bigl\{dist(x, \mathcal{S}) : x \in \tilde{\mathcal{S}}\bigr\} < \epsilon$,
(3) *if $x \in |P|$ and $dist(x, \mathcal{S}) \geq \epsilon$ then $\tilde{\Phi}(x) = \Phi(x)$,*
(4) *If $\sigma \in P$ then $\tilde{\Phi}(\sigma \setminus \tilde{\mathcal{S}}) \subset \Phi(\sigma \setminus \mathcal{S})$, and*
(5) $\mathcal{H}^a(\tilde{\mathcal{S}}) < K\mathcal{H}^a(\mathcal{S})$.

Remark 1.6. A "cell" is a closed, bounded region of some Euclidean space defined by finitely many linear equalities and inequalities. Theorem 1.1 probably can be generalized to general finite "cell complexes" (Munkres [Mun66, Definition 7.6, p. 74]), i.e., complexes consisting of cells that fit together nicely. A "cubical set" (Kaczynski *et al* [KMM04, Definition 2.9, p. 43]) is an example. Since any cell complex has a finite simplicial subdivision (Munkres [Mun66, Lemma 7.8, p. 75]), corollary 1.5 certainly extends immediately to finite cell complexes.

Remark 1.7. The set of applications of theorem 1.1 is nonempty. It turns out that multivariate statistical procedures often have "singularities", i.e., data sets at which the procedure, regarded as a function, does not have a limit. (See e.g., [Ell91, Ell02, Ell03, Ell04, Ellb].) Let $\mathcal{S}' \subset |P|$ be the singular set (set of singularities) of a data analytic procedure $\Psi : |P| \setminus \mathcal{S}' \to \mathsf{F}$. $\mathcal{S}'$ may not be closed, so we cannot apply the theorem with $\mathcal{S} = \mathcal{S}'$. However, it turns out that Ψ can often be replaced by another procedure $\Phi : |P| \setminus \mathcal{S} \to \mathsf{F}$, where $\mathcal{S}$ is a closed subset of $\mathcal{S}'$ consisting of the most "severe", and therefore most interesting, singularities of Φ. It frequently turns out that $\dim \mathcal{S}$ is bounded below by some integer a depending on general features of the statistical problem. If $\dim \mathcal{S} > a$, then $\mathcal{H}^a(\mathcal{S}') \geq \mathcal{H}^a(\mathcal{S}) = \infty$. Assume $\dim \mathcal{S} = a$ and apply theorem 1.1 to Φ. It turns out that the set $\tilde{\mathcal{S}}$ must also have Hausdorff dimension a. Hence, by remark 1.4(3), the $\mathcal{H}^a$-volume of $\tilde{\mathcal{S}}$, and, hence, of $\mathcal{S}$ and $\mathcal{S}'$, is bounded below. The paper [Ellb] (in preparation) will develop and refine this idea.

Remark 1.8. We make no effort to compute the best, or even a good, constant K in (1.1). In principle, for a given P and a one could follow the proof and compute some value of K, but it would probably be very large. Let $K_P(a)$ denote the best, i.e., smallest, possible value of K in (1.1). It would be helpful and interesting to know something about the relationship between K_P and the structure of P.

Now we briefly discuss the key ideas of the proof. Let $\sigma \in Q$ be a simplex of dimension $> a$. Let $\mathcal{A} = \mathcal{S} \cap \sigma$. If $\operatorname{Int} \sigma \subset \mathcal{A}$ or $(\operatorname{Int} \sigma) \cap \mathcal{A} = \varnothing$, then nothing much has to be done in $\operatorname{Int} \sigma$. So suppose $\varnothing \neq (\operatorname{Int} \sigma) \cap \mathcal{A} \neq \operatorname{Int} \sigma$. Call such a $\sigma \in Q$ a "partial simplex" (of $\mathcal{S}$). The process of deforming $\mathcal{S}$ so that it becomes a subcomplex involves "pushing $\mathcal{A}$ out" of $\operatorname{Int} \sigma$ from a point $z \in (\operatorname{Int} \sigma) \setminus \mathcal{A}$. Figure 1 illustrates this. Given a point $z \in (\operatorname{Int} \sigma) \setminus \mathcal{A}$, move every point, $y \in \mathcal{A}$ along the ray emanating from z out to a point $\bar{h}_{z,\sigma}(y) \in \operatorname{Bd} \sigma$. This results in a new $\mathcal{S}$, call it $\mathcal{S}'$. If the dimension of σ is maximal among all partial simplices in Q, we have

$$\mathcal{S}' = (\mathcal{S} \setminus \sigma) \cup \bar{h}_{z,\sigma}(\mathcal{A}) \tag{1.4}$$

(lemma 2.1(a, d, f, h)).

The $\mathcal{H}^a$-volume of the image of $(\operatorname{Int} \sigma) \cap \mathcal{A}$ as it is flattened against the sides of the simplex can be larger or smaller than $\mathcal{H}^a((\operatorname{Int} \sigma) \cap \mathcal{A})$ (figure 1). One can easily imagine a subset $\mathcal{A}$ of the simplex, σ, in figure 1 with the property that the image of $\mathcal{A}$ after pushing out from *any* interior point z of σ would have large 1-dimensional Hausdorff measure. But in order to have this property the set $\mathcal{A}$ itself must have large 1-dimensional Hausdorff measure. So that observation does not lead to a counter example to the theorem.

Call the ratio of image volume, $\mathcal{H}^a\big[\bar{h}_{z,\sigma}(\mathcal{A})\big]$, to input volume, $\mathcal{H}^a(\mathcal{A})$, the "$(\sigma, \mathcal{A}, z)$-magnification factor." In order to prove (1.1) we need to always choose $z = z_0(\sigma, \mathcal{S}) \in \operatorname{Int} \sigma$ so that the magnification factor is bounded independently of $\mathcal{A}$. The existence of such an ($\mathcal{A}$-dependent) z is shown by demonstrating that, averaged over $z \in \operatorname{Int} \sigma$, the magnification factor is bounded independently of $\mathcal{A}$. (Actually, we average over z in a concentric simplex sitting in the interior of σ.) The bound turns out to not depend on the size of σ, but only on its shape, specifically, its "thickness" (appendix B). That is important because one can arbitrarily finely subdivide a finite simplicial complex all the while controlling the thickness of the simplices.

The operation of pushing $\mathcal{S}$ out of simplices is performed recursively. Let $\mathcal{S}^0 = \mathcal{S}$ and $\Phi_0 = \Phi$. Push $\mathcal{S}^0$ out of a partial simplex $\sigma \in Q$ of highest dimension. The pushing out operation results in a new set, *viz.*, $\mathcal{S}'$ as defined by (1.4) (with $\mathcal{S}^0$ in place of $\mathcal{S}$). A new function, call it Φ', must be defined that is continuous off $\mathcal{S}'$.

Let $\rho \in P$ and suppose σ is a face of ρ (e.g., $\rho = \sigma$). If $\dim \rho > \dim \sigma$, then by maximality of $\dim \sigma$, the simplex ρ will not be a partial simplex of $\mathcal{S}^0$. Thus, either $(\operatorname{Int} \rho) \cap \mathcal{S}^0 = \varnothing$ or $\operatorname{Int} \rho \subset \mathcal{S}^0$. If $\operatorname{Int} \rho \subset \mathcal{S}^0$ then $\rho \subset \mathcal{S}^0$, since $\mathcal{S}^0$ is closed. Therefore, $\sigma \subset \mathcal{S}^0$, since $\sigma \subset \rho$. But σ is a partial simplex of $\mathcal{S}$. Therefore, $(\operatorname{Int} \rho) \cap \mathcal{S}^0 = \varnothing$. The map Φ has to be deformed in ρ (by composing it on the right with a locally Lipschitz map $g : \rho \setminus \bar{h}_{z,\sigma}(\mathcal{A}) \to \rho$) so that the resulting map, call it Φ', is still defined and continuous in $\operatorname{Int} \rho$. Now let $\mathcal{S}^0 = \mathcal{S}'$ and $\Phi_0 = \Phi'$. Repeat until no partial simplices remain.

Remark 1.9. $\tilde{\Phi}$ and $\tilde{\mathcal{S}}$ are probably not constructible unless the set $\mathcal{S}$ is "decidable" (Blum *et al* [BCSS98, Definition 2, p. 47]).

Remark 1.10 ("Tomography"). By part (9) of theorem 1.1, $\mathcal{H}^a\big(\tilde{\mathcal{S}} \cap |Q|\big)$ gives some information about $\mathcal{H}^a\big(\mathcal{S} \cap |Q|\big)$. Here we show that $\tilde{\mathcal{S}}$ is a deformation retract of a set determinted by $\mathcal{S}$.

Recall that $\tilde{\mathcal{S}}$ is constructed by recursively pushing the compact sets $\mathcal{S}^0$ from the simplicial interiors of its partial simplices of maximal dimension. Track the recursive construction of $\tilde{\mathcal{S}}$ by letting $\mathcal{S}_i$ denote the set obtained after i pushing operations. Thus, $\mathcal{S}_0 = \mathcal{S}$. For some value $i = m$, the set $\mathcal{S}_i \cap |Q|$ will be the underlying space of a subcomplex of Q. Then $\tilde{\mathcal{S}} = \mathcal{S}_m$. The recursive construction thus stops after m steps. Unless otherwise specified let $i = 0, \ldots, m$.

If $\sigma \in Q$ then for some values of i, the simplex σ may be a partial simplex of $\mathcal{S}_i$ but for i sufficiently large, σ will not be a partial simplex (lemma 2.1, points (a, d, h)). Let σ_i be the partial simplex of $\mathcal{S}_i$ from which $\mathcal{S}_i$ will be pushed to produce $\mathcal{S}_{i+1}$. Note that for any $\sigma \in P$, there is at most one i s.t. $\sigma = \sigma_i$. There is a point

$$z_{0,i} \in \operatorname{Int} \sigma_i \setminus \mathcal{S}_i \tag{1.5}$$

from which $\mathcal{S}_i$ will be pushed. If $0 \leq i < m$ and $y \in \operatorname{Int} \sigma_i$, let $h_i(y) = \bar{h}_{z_{0,i},\sigma_i}(y)$. If $y \in |Q| \setminus (\operatorname{Int} \sigma_i)$ or if $y \in |Q|$ and $i \geq m$, let $h_i(y) = y$. Note that, by (1.4),

$$h_i(\mathcal{S}_i) = \mathcal{S}_{i+1}. \tag{1.6}$$

For each $i = 0, \ldots, m$, the function h_i is continuous on $|Q| \setminus \{z_{0,i}\}$. This is because, first, h_i is trivially continuous on $|Q| \setminus (\operatorname{Int} \sigma_i)$. Second, by lemma A.2 part (7), h_i is continuous on $\sigma_i \setminus \{z_{0,i}\}$. Finally, by (2.9), $\bar{h}_{z_{0,i}}(y) = y = h_i(y)$ for $y \in \operatorname{Bd} \sigma_i$.

For $i = 0, \ldots, m-1$, let $L_i(y)$ be the, possibly trivial, closed line segment joining $y \in \mathcal{S}_i$ to $h_i(y) \in \mathcal{S}_{i+1}$. So $L_i(y) \subset |Q|$. Notice that for $y \in \mathcal{S}_i$,

$$L_i(y) \cap (\operatorname{Bd} \sigma_i) = \{h_i(y)\} \text{ or } L_i(y) \cap (\operatorname{Bd} \sigma_i) = \varnothing. \tag{1.7}$$

Observe also

$$\begin{aligned} &\text{If } y \in \mathcal{S}_i \text{ then } z_{0,i} \notin L_i(y). \\ &\text{If } y \in \mathcal{S}_i \text{ and } y' \in L_i(y) \text{ then } h_i(y') = h_i(y). \end{aligned} \tag{1.8}$$

Now let

$$\mathcal{D}_i := \bigcup_{y \in \mathcal{S}_i} L_i(y) \text{ and } \mathcal{E}_i := \bigcup_{j=i}^{m} \mathcal{D}_j, \quad i = 0, \ldots, m-1. \text{ Let } \mathcal{E}_m = \mathcal{S}_m = \tilde{\mathcal{S}}.$$

Note that $\mathcal{S} \subset \mathcal{E}_0$. *Claim:* For $i = 0, \ldots, m-1$, the set $\mathcal{E}_{i+1}$ is a deformation retract of $\mathcal{E}_i$. Let $F_i : \mathcal{E}_i \times [0,1] \to \mathcal{E}_i$ be defined by

$$F_i(y,t) = \begin{cases} y, & \text{if } y \in \mathcal{E}_{i+1}, \\ (1-t)y + t\, h_i(y), & \text{if } y \in \mathcal{E}_i \setminus \mathcal{E}_{i+1}\ . \end{cases}$$

We show that F_i is a deformation retraction. First, we show that $F_i\big(\mathcal{E}_i \times [0,1]\big) \subset \mathcal{E}_i$. It suffices to show that

$$F_i\Big(\big[\mathcal{E}_i \setminus \mathcal{E}_{i+1}\big] \times [0,1]\Big) \subset \mathcal{E}_i. \tag{1.9}$$

Let $t \in [0,1]$ and $y \in \mathcal{E}_i \setminus \mathcal{E}_{i+1}$. Then there exists $y' \in \mathcal{S}_i$ s.t. $y \in L_i(y')$. By (1.8), we have $h_i(y) = h_i(y')$. Therefore, $(1-t)y + t\, h_i(y) \in L_i(y') \subset \mathcal{E}_{i+1}$. The equation $h_i(y) = h_i(y')$ and (1.6) also implies that

$$h_i(\mathcal{E}_i) \subset \mathcal{E}_{i+1}. \tag{1.10}$$

This completes the proof of (1.9). Next, observe that F_i is continuous. By (1.10), we have $F_i(y,1) \in \mathcal{E}_{i+1}$ for every $y \in \mathcal{E}_i$. Finally, obviously $F_i(y,0) = y$ and $F_i(y,t) = y$ for every $y \in \mathcal{E}_{i+1}$.

Obviously, by first retracting $\mathcal{E}_0$ onto $\mathcal{E}_1$ then $\mathcal{E}_1$ onto $\mathcal{E}_2$ and so forth, finally retracting $\mathcal{E}_{m-1}$ onto $\mathcal{E}_m$, the net result is a deformation retraction of $\mathcal{E}_0$ onto $\mathcal{E}_m = \tilde{\mathcal{S}}$. Hence, the homology of $\tilde{\mathcal{S}}$ is the same as that of $\mathcal{E}_1$. Thus, if one is willing to accept $\mathcal{E}_1$ as some sort of approximation

to $\mathcal{S}$ then the homology of $\tilde{\mathcal{S}}$ should be considered an approximation to the homology of $\mathcal{S}$. However, in fact, the relationship between the homologies of $\mathcal{S}$ and $\tilde{\mathcal{S}}$ is very loose at best.

It follows from the preceding that $\mathcal{S}$ is deformable in $|P|$ into (onto, actually) $\tilde{\mathcal{S}}$ (Spanier [Spa66, p. 29]).

Morever, note that the approximation $\mathcal{E}_1$ is not natural since it depends on the somewhat arbitrary choices of the $z_{0,i}$'s.

Remark 1.11. Theorem 1.1 is reminiscent of the Deformation Theorem (Federer [Fed69, pp. 401 – 408], Simon [Sim83, 29.1, p. 163 and 29.4, p. 166], Hardt and Simon [HS86, Hardt's Lecture 3, pp. 83–93] and Giaquinta *et al* [GMS98, Lemma 2, p. 495; Theorem 1, p. 498; and Theorem 2, p. 503, Volume I]) in geometric measure theory. E.g., in the proof of the Deformation Theorem as in the proof of theorem 1.1, a set, $\mathcal{S}$, also gets deformed by pushing it out of the interior of cells onto the boundaries. In theorem 1.1 the cells are simplicial while in the Deformation Theorem the cells are cubical, but that is not important.

Might theorem 1.1 follow from the Deformation Theorem? Here I argue that deducing theorem 1.1 from the Deformation Theorem would not be straight forward. The Deformation Theorem shows that a "normal current" can be approximated by one supported by a cubical complex. (See preceding references for relevant terminology from geometric measure theory.) How might we show that, at least, a "cubical" version of theorem 1.1 follows from the Deformation Theorem? Consider the case where $a \in (0, p)$ is an integer. We may assume that P sits in a finite dimensional Euclidean space, $\mathbb{R}^N$.

In order to apply the Deformation Theorem to T we must assume T is a normal current, i.e., $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$, where $\mathbf{M}$ denotes "mass" (Giaquinta *et al* [GMS98, p. 125, Volume I]). A natural way to make $\mathcal{S}$ into a normal current, T, is to let T be a current of the form

$$T(\omega) = \int_{\mathcal{S}} \langle \omega(x), \vec{T}(x) \rangle \, d\mu(x) \quad \big(\omega \in \mathcal{D}^a(\mathbb{R}^N)\big),$$

where μ is a Radon measure on $\mathbb{R}^N$ (e.g., $\mu = \mathcal{H}^a$), $\vec{T}$ is an $\mathcal{H}^a$-measurable function taking values in the space of a-vectors of length 1, $\mathbb{D}^a(\mathbb{R}^n)$ is the space of C^∞ differential a-forms on $\mathbb{R}^N$ with compact support, and "$\langle \cdot, \cdot \rangle$" indicates evaluation of the first argument at the second. (See Giaquinta *et al* [GMS98, Theorem 1, p. 126, Volume I] and Hardt and Simon [HS86, p. 67].) To avoid trivialities assume $\mu(\mathcal{S}) > 0$. Then

$$T \text{ is not } 0.$$

For concreteness, suppose P is a 2-simplex, σ, (and its faces and vertices) sitting in $\mathbb{R}^2$ and $a = 1$. Suppose $\mathcal{S} \subset \operatorname{Int} \sigma$ with $\dim \mathcal{S} = a = 1$. (In particular, $\mathcal{S}$ has empty interior as a subset of P.) Since $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$, by Giaquinta *et al* [GMS98, Theorem 2, p. 129, Volume I], Simon [Sim83, Lemma 26.29, pp. 143], or Hardt and Simon [HS86, Theorem 2.4, p. 78], if L and $L^\perp$ are two perpendicular one-dimensional subspaces of $\mathbb{R}^2$, then, since $T \neq 0$, the projections of $\mathcal{S}$ onto L and $L^\perp$ cannot both have 0 Lebesgue measure.

But suppose $\mathcal{S} \subset \mathbb{R}^2$ is a compact "Cantor dust" of dimension 1 and having positive $\mathcal{H}^1$ measure (Falconer [Fal90, pp. xvi, 31]). But there exists perpendicular lines L and $L^\perp$ s.t. the projections of $\mathcal{S}$ onto them each have Lebesgue measure 0. (Falconer [Fal90, Example 6.7, p. 87]). So the Deformation Theorem does not apply to T.

If theorem 1.1 is not a consequence of the Deformation Theorem itself, perhaps the method of proof of the Deformation Theorem might be used to prove theorem 1.1. But at least some

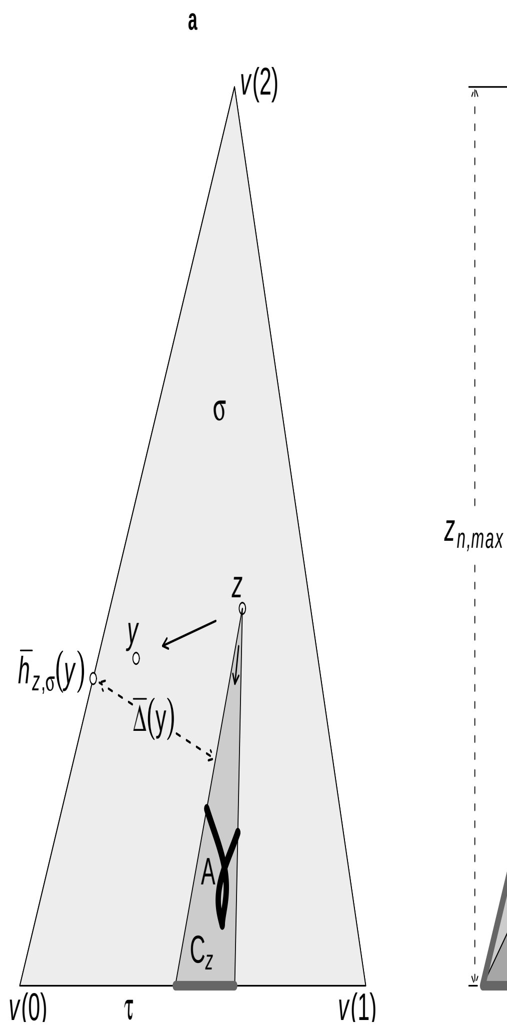

Figure 1. Pushing $\mathcal{S}$ out of a simplex, σ (light grey triangles). Heavy black curves constitute possible $\mathcal{A}$'s. (But $\mathcal{A}$ can be any compact subset of σ.) Heavy dark grey lines are possible images, $\bar{h}_{z,\sigma}(\mathcal{A})$, of $\mathcal{A}$ after pushing it out of σ from z. (a) Sometimes pushing reduces volume. Medium grey triangle is the set $\mathcal{C}_z$. $v(0)$, $v(1)$, and $v(2)$ are the vertices of σ. τ is the 1-simplex that is the "bottom" face of σ. y is a generic point of σ. Its image under $\bar{h}_{z,\sigma}$ is obtained by pushing it along the ray emanating from z out to the boundary. $\bar{\Delta}(y)$ is the distance from that image to $\mathcal{C}_z$. (b) The worrisome case is when pushing increases volume. $\zeta(z,\tau)$ is the medium dark grey triangle. $\mathcal{C}_z$ is the union of the medium grey regions and $\zeta(z,\tau)$. $z_{n,max}$ is the height of σ from the plane spanned by τ. Points of $\mathcal{A}$ on $\mathrm{Bd}\,\sigma$ are unaffected by pushing.

proofs of the Deformation Theorem rely on "slicing" (Giaquinta *et al* [GMS98, pp. 151–152, Volume I]). It is not clear how to create a useful analogue of slicing that applies to general compact sets.

2. Proof of theorem 1.1

2.1. **Pushing out.** We may assume that P sits in a finite dimensional Euclidean space, $\mathbb{R}^N$. The metric on $|P|$ is the one it inherits from the Euclidean space. Let $\sigma \in Q$ be an n-simplex with $n > 0$. By definition of simplex (appendix B), σ does not lie in any $(n-1)$-dimensional affine plane. Let $\mathcal{S} \subset |P|$ be closed and suppose $\dim\bigl(\mathcal{S} \cap |Q|\bigr) \leq a$. Suppose

$$\operatorname{Int}\sigma \nsubseteq \mathcal{S} \text{ but } (\operatorname{Int}\sigma) \cap \mathcal{S} \neq \varnothing. \tag{2.1}$$

I.e., σ is a "partial simplex" of $\mathcal{S}$. We wish to redefine Φ so that it is continuous in $\operatorname{Int}\sigma$. (Of course, it may be possible to continuously extend the *restriction*, $\Phi|_{\operatorname{Int}\sigma}$, of Φ to all of $\operatorname{Int}\sigma$. But it may not be possible to extend Φ itself to be continuous on $\operatorname{Int}\sigma$ because of the behavior of Φ on $|P| \setminus \sigma$.) Consider the following construction. Let

$$\mathcal{A} = \mathcal{S} \cap \sigma. \tag{2.2}$$

So by (2.1) and compactness of $\mathcal{S}$,

$$\mathcal{A} \text{ is compact and } \varnothing \neq \mathcal{A} \cap (\operatorname{Int}\sigma) \neq \operatorname{Int}\sigma. \tag{2.3}$$

Let

$$z \in (\operatorname{Int}\sigma) \setminus \mathcal{A} = (\operatorname{Int}\sigma) \setminus \mathcal{S} \tag{2.4}$$

be fixed. Let $\mathcal{C} \subset \sigma$ be the set

$$\mathcal{C} := \mathcal{C}_z := \mathcal{C}_z(\sigma) := \mathcal{C}_z(\sigma, \mathcal{S}) := \{\alpha x + (1-\alpha)z \in \sigma : x \in \mathcal{A} \text{ and } \alpha \geq 0\}. \tag{2.5}$$

(Thus, $\alpha > 1$ is possible. See figure 1.) $\mathcal{C}$ consists of the intersection of σ with the union of all rays emanating from z and passing through some point of $\mathcal{A}$. *Claim:*

$$\mathcal{C} \text{ is compact, } \mathcal{S} \cap \sigma = \mathcal{A} \subset \mathcal{C}, \text{ and } z \in \mathcal{C} \setminus \mathcal{A}. \tag{2.6}$$

It suffices to show that $\mathcal{C}$ is closed. Let $\{y_m\} \subset \mathcal{C}$ and suppose $y_m \to y \in \sigma$. Write

$$y_m = \alpha_m x_m + (1-\alpha_m)z = \alpha_m(x_m - z) + z, \quad \alpha_m \geq 0,\ x_m \in \mathcal{A}, \text{ and } m = 1, 2, \ldots. \tag{2.7}$$

Now, $z \notin \mathcal{A}$ and $\mathcal{A}$ is compact. Therefore, there exists $\delta > 0$ s.t. $|x_m - z| \geq \delta$. Hence, by (2.7),

$$0 \leq \alpha_m \leq \frac{diam(\sigma)}{\delta} < \infty.$$

Therefore, by compactness, extracting a subsequence if necessary, we have $\alpha_m \to \alpha \geq 0$ and $x_m \to x \in \mathcal{A}$ so

$$y_m = \alpha_m x_m + (1-\alpha_m)z \to \alpha x + (1-\alpha)z \in \mathcal{C}.$$

This proves the claim (2.6).

If $x \in \sigma$, choose $b(x) = b_{z,\sigma}(x) \in [0,1]$ and $\bar{h}(x) = \bar{h}_\sigma(x) = \bar{h}_{z,\sigma}(x) \in \operatorname{Bd}\sigma$ s.t.

$$\bar{h}(x) + \bigl(1 - b(x)\bigr)\bigl(z - \bar{h}(x)\bigr) = b(x)\bigl(\bar{h}(x) - z\bigr) + z = b(x)\bar{h}(x) + \bigl(1 - b(x)\bigr)z = x. \tag{2.8}$$

(See figure 1.) If $x \neq z$ then $b(x)$ and $\bar{h}(x)$ are unique (lemma B.1; define $\bar{h}(z) \in \operatorname{Bd}\sigma$ arbitrarily). If $x = z$, then $b(x)$ is still unique because $z \in \operatorname{Int}\sigma$. Note that

$$\bar{h}(x) = x \text{ if and only if } x \in \operatorname{Bd}\sigma. \tag{2.9}$$

and

$$b(x) = 1 \text{ if and only if } x \in \operatorname{Bd}\sigma \text{ and } b(x) = 0 \text{ if and only if } x = z. \tag{2.10}$$

Claim:

$$\mathcal{C} \cap (\operatorname{Bd}\sigma) = \bar{h}(\mathcal{A}). \tag{2.11}$$

Suppose $x \in \sigma \setminus \{z\}$. Then

$$b(x)\bar{h}(x) + \big(1 - b(x)\big)z = x \text{ and } b(x) \in (0, 1].$$

Then

$$\begin{aligned}\bar{h}(x) &= b(x)^{-1}x + b(x)^{-1}\big(b(x) - 1\big)z \\ &= b(x)^{-1}x + \big(1 - b(x)^{-1}\big)z.\end{aligned}$$

Hence, if $x \in \mathcal{A}$, then $\bar{h}(x) \in \mathcal{C} \cap (\operatorname{Bd}\sigma)$ by definition.

Conversely, suppose $x \in \mathcal{C} \cap (\operatorname{Bd}\sigma)$. Then there exists $\alpha > 0$ and $y \in \mathcal{A}$ (so by (2.4) $y \neq z$) s.t.

$$x = \alpha y + (1 - \alpha)z. \tag{2.12}$$

Consider the ray

$$R(t) := ty + (1 - t)z = y + (t - 1)(y - z), \quad t \geq 0.$$

Then $R(0) = z$ and $R(1) = y \in \sigma$ and for $t > 1$ sufficiently large we have $R(t) \notin \sigma$. By lemma B.1 there exists a unique $t_0 > 0$ s.t. $R(t_0) \in \operatorname{Bd}\sigma$. But if $0 < t < 1$, then $R(t) \in \operatorname{Int}\sigma$. Therefore, we must have $\alpha = t_0 \geq 1$. From (2.12) we get

$$y = \alpha^{-1}x + (1 - \alpha^{-1})z.$$

I.e., $x = \bar{h}(y)$ and $b(y) = \alpha^{-1} \in [0, 1]$. This proves the claim (2.11).

Moreover, we *claim*

$$\text{For } y \in \sigma \setminus \{z\} \text{ we have } y \in \mathcal{C} \text{ if and only if } \bar{h}(y) \in \mathcal{C}. \tag{2.13}$$

To see this, first assume $\bar{h}(y) \in \mathcal{C}$. Then, by (2.11), $\bar{h}(y) = \bar{h}(x)$ for some $x \in \mathcal{A}$. Now, $x \neq z$ (by (2.4)) so $b(x) > 0$ by (2.10). Hence, letting $\alpha = 1/b(x)$, we have

$$x = \alpha^{-1}\bar{h}(y) + (1 - \alpha^{-1})z \text{ and } y = b(y)\bar{h}(y) + \big[1 - b(y)\big]z.$$

Thus, $\bar{h}(y) = \alpha x + (1 - \alpha)z$ so

$$y = b(y)\alpha x + b(y)(1 - \alpha)z + \big[1 - b(y)\big]z = b(y)\alpha x + \big[1 - b(y)\alpha\big]z.$$

Thus, $y \in \mathcal{C}$.

Conversely, suppose $y \in \mathcal{C} \setminus \{z\}$. Then $b(y) > 0$ by (2.10) and there exists $x \in \mathcal{A}$ and $\alpha \geq 0$ s.t.

$$b(y)\bar{h}(y) + \big[1 - b(y)\big]z = y = \alpha x + (1 - \alpha)z.$$

Thus,

$$\bar{h}(y) = b(y)^{-1}\alpha x + \big(1 - b(y)^{-1}\alpha\big)z.$$

Thus, $\bar{h}(y) \in \mathcal{C}$. This completes the proof of the claim (2.13).

Here and below we find it convenient to restrict ourselves to the situation when $\bar{h}(x)$ belongs to a specified proper face of σ. Let τ be a face of σ of dimension $n-1$. Without loss of generality (WLOG) we may assume

$$\sigma \subset \mathbb{R}^n \text{ and } \tau \subset \mathbb{R}^{n-1}, \tag{2.14}$$

where we identify $\mathbb{R}^{n-1}$ with $\{(y_1, \ldots, y_{n-1}, 0) \in \mathbb{R}^n : (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}\}$. (Recall $n = \dim \sigma$.) Thus, the n^{th} coordinate of any $y \in \sigma \setminus \tau$ is non-zero. If $y \in \sigma \setminus \{z\}$ let

$$\begin{aligned} b = b(y) = b(y; z, \tau) \text{ and } h = h_{z,\tau}(y) = h(y; z, \tau) \text{ be the solution of } bh + (1-b)z = y \\ \text{with } b \in (0, 1] \text{ and } h \in \tau, \text{ if such } b(y) \text{ and } h_{z,\tau}(y) \text{ exist.} \end{aligned} \tag{2.15}$$

Otherwise, $b(y; z, \tau)$ and $h(y; z, \tau)$ are undefined. Thus,

$$b(y; z, \tau) = b_{z,\sigma}(y) \text{ and } h(y; z, \tau) = \bar{h}_{z,\sigma}(y), \text{ if } y \in \sigma \setminus \{z\} \text{ and } \bar{h}_{z,\tau}(y) \in \tau. \tag{2.16}$$

The set of $y \in \sigma$ for which $b(y)$ and $h_{z,\tau}(y)$ exist uniquely is obviously precisely the set

$$\zeta(z) = \zeta(z; \tau) = \{bx + (1-b)z \in \sigma : x \in \tau,\ 0 < b \leq 1\}. \tag{2.17}$$

(See figure 1.) We have

$$\bar{h}_{z,\sigma}(y) = h_{z,\tau}(y), \qquad \text{for } y \in \zeta(z). \tag{2.18}$$

In particular,

$$\bar{h}_{z,\sigma}\big(\mathcal{A} \cap (\operatorname{Int} \sigma)\big) \cap \tau = h_{z,\tau}\Big(\mathcal{A} \cap \big[(\operatorname{Int} \sigma) \cap \zeta(z; \tau)\big]\Big) \text{ for } z \in (\operatorname{Int} \sigma) \setminus \mathcal{A}. \tag{2.19}$$

In particular, if τ' is another $(n-1)$-face of σ that intersects τ then (2.19) implies

$$\zeta(z; \tau) \cap \zeta(z; \tau') \neq \varnothing \text{ and } h_{z,\tau}(y) = h_{z,\tau'}(y) \in \tau \cap \tau' \text{ for } y \in \zeta(z; \tau) \cap \zeta(z; \tau'). \tag{2.20}$$

Notice further that

$$h_{z,\tau}(y) = y, \text{ if } y \in \tau. \tag{2.21}$$

Note that $z \notin \zeta(z; \tau)$. We *claim*

$$\sigma \setminus \{z\} = \bigcup_{\omega} \zeta(z; \omega), \tag{2.22}$$

where the union is taken over all $(n-1)$-faces, ω, of σ. For let $y \in \sigma \setminus \{z\}$ and write

$$y = \sum_{i=0}^{n} \beta_i v(i) \text{ and } z = \sum_{i=0}^{n} \alpha_i v(i),$$

where $v(0), \ldots, v(n) \in \mathbb{R}^n$ are the vertices of σ. Then all the β_i's (α_i's) are nonnegative and sum to 1. Since $z \in \operatorname{Int} \sigma$ the coordinates $\alpha_0, \ldots, \alpha_n$ are all strictly positive. Let $i = j$ maximize $(\alpha_i - \beta_i)/\alpha_i$. (Define $(\alpha_i - \beta_i)/\alpha_i = -\infty$ if $\alpha_i = 0$.) Since $\sum_i \alpha_i = 1 = \sum_i \beta_i$, but $(\alpha_0, \ldots, \alpha_n) \neq (\beta_0, \ldots, \beta_n)$ we have $\alpha_i > \beta_i$ for at least one i. Therefore, we have $\alpha_j > \beta_j > 0$ and $b := (\alpha_j - \beta_j)/\alpha_j > 0$. For $i = 0, \ldots, n$, let

$$\gamma_i = \frac{\alpha_j \beta_i - \alpha_i \beta_j}{\alpha_j - \beta_j}.$$

Now for $i = 0, \ldots, n$,

$$(\alpha_j - \beta_j)\gamma_i = \alpha_i(\alpha_j - \beta_j) - \alpha_j(\alpha_i - \beta_i) \geq 0$$

by choice of j. I.e., since $\alpha_j > \beta_j$, we have $\gamma_i \geq 0$ for $i = 0, \ldots, n$. Moreover, $\sum_i \gamma_i = 1$. Hence, if $x = \sum_{i=0}^n \gamma_i v(i)$, then $x \in \sigma$. But $\gamma_j = 0$, so if ω is the $(n-1)$-face of σ opposite $v(j)$, then actually $x \in \omega$. But $b \in (0, 1]$ and it is easy to see that $y = bx + (1-b)z$. I.e., $y \in \zeta(z; \omega)$. This proves the claim (2.22).

(Thus, $b(y) = (\alpha_j - \beta_j)/\alpha_j > 0$ if $y \neq z$. But $b(y) = (\alpha_j - \beta_j)/\alpha_j = 0$ still works if $y = z$. Hence, it follows from lemma B.2, (C.8), and the fact that "max" is Lipschitz that $b(y)$ is Lipschitz in $y \in \sigma$.)

If $y \in \zeta(z)$ then obviously

$$h_{z,\tau}(y) = b(y)^{-1}(y - z) + z. \tag{2.23}$$

If $y \in \mathbb{R}^n$, let y_j be the j^{th} coordinate of y $(j = 1, \ldots, n)$. Since $\tau \subset \mathbb{R}^{n-1}$, $x_n = 0$ if $x \in \tau$. If $v(n)$ is the vertex of σ opposite τ then its n^{th} coordinate, $v_n(n)$, is not 0. WLOG we may assume $v_n(n) > 0$. Then $y \in \sigma$ implies $y_n \geq 0$. Since $z \in \operatorname{Int} \sigma$, we have $z_n > 0$. Therefore,

$$y \in \zeta(z) \text{ implies } 0 \leq y_n < z_n. \tag{2.24}$$

Then (2.23) implies

$$0 = b(y)^{-1}(y_n - z_n) + z_n.$$

Thus,

$$b(y) = z_n^{-1}(z_n - y_n). \tag{2.25}$$

In particular, by (2.15),

$$y_n \uparrow z_n \text{ with } y \in \zeta(z) \text{ implies } b(y) \to 0, \text{ which implies } y \to z. \tag{2.26}$$

Substituting (2.25) into (2.23) we get

$$h_{z,\tau}(y) = \frac{z_n}{z_n - y_n}(y - z) + z = \frac{z_n}{z_n - y_n}\left(y - \frac{y_n}{z_n} z\right), \quad y \in \zeta(z). \tag{2.27}$$

Thus, $h_{z,\tau}$ is continuous on $\zeta(z)$. Conversely, we have

$$\text{If } y \in \sigma,\ y_n < z_n, \text{ and } \frac{z_n}{z_n - y_n}(y - z) + z \in \tau, \text{ then } y \in \zeta(z). \tag{2.28}$$

Since $\mathcal{A}$ is compact and $z \notin \mathcal{A}$, by (2.4) and (2.26), the difference $z_n - y_n > 0$, $y \in \zeta(z) \cap \mathcal{A}$, is bounded away from 0. It follows from corollary C.4 in appendix C that

$$h_{z,\tau} \text{ is locally Lipschitz on } \zeta(z) \text{ and Lipschitz on } \mathcal{A} \cap \zeta(z). \tag{2.29}$$

(Lemma A.2, part (7), will tell us that $\bar{h} : \sigma \setminus \{z\} \to \operatorname{Bd} \sigma$ is locally Lipschitz on $\sigma \setminus \{z\}$ and Lipschitz on $\mathcal{A}$.)

For $y \in \sigma \setminus \{z\}$, let

$$\bar{\Delta}(y) := \operatorname{dist}\bigl(\bar{h}(y), \mathcal{C}\bigr). \tag{2.30}$$

(See figure 1.) Note that $\bar{\Delta}(y) \in \bigl[0, diam(\sigma)\bigr]$ for all $y \in \sigma \setminus \{z\}$. Define.

$$\bar{\Delta}(z) := diam(\sigma) > 0 \tag{2.31}$$

and define $s : \sigma \times [0, 1] \to \sigma$ as follows

Let k be the function

$$k(\beta, \delta, t) := \begin{cases} \exp\left\{-\frac{\delta + t}{1 - \beta} + \delta + t\right\} = \exp\left\{-\frac{\beta(\delta + t)}{1 - \beta}\right\}, & \text{if } \delta, t \in \mathbb{R}, \beta < 1, \\ 0, & \text{if } \delta + t > 0 \text{ and } \beta = 1. \end{cases} \tag{2.32}$$

Let $y \in \sigma$, $0 \leq t \leq 1$, write $b(y) = b_{z,\sigma}(y)$, and let

$$f(y,t) := k\big[b(y), \bar{\Delta}(y), t\big], \text{ for } (y,t) \in B_z := \big(\sigma \times [0,1]\big) \setminus \Big[\big(\mathcal{C}_z \cap [\mathrm{Bd}\,\sigma]\big) \times \{0\}\Big]. \tag{2.33}$$

Finally, let

$$s(y,t) := s_z(y,t) := \begin{cases} y, & \text{if } b(y) = 1, \text{ i.e. } y \in \mathrm{Bd}\,\sigma, \\ \big(1 - f(y,t)\big)\bar{h}(y) + f(y,t)\, z, & \text{if } 0 \leq b(y) < 1, \text{ i.e., } y \in \mathrm{Int}\,\sigma. \end{cases} \tag{2.34}$$

If $y \in \overline{\mathrm{St}}\,\sigma$ (see appendix B), we can write

$$y = \mu(y)\sigma(y) + \big(1 - \mu(y)\big)w(y),$$

where $\mu(y) \in [0,1]$, $\sigma(y) \in \sigma$, and $w(y) \in \mathrm{Lk}\,\sigma$. (See proof of lemma 2.1 below for details.)

Next, we define $g : |P| \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big] \to |P|$. If $\overline{\mathrm{St}}\,\sigma \neq \sigma$ define

$$g(y) := g_z(y) := \tag{2.35}$$

$$\begin{cases} \mu(y)\, s_z\big(\sigma(y), 1 - \mu(y)\big) + \big(1 - \mu(y)\big)w(y) \in \overline{\mathrm{St}}\,\sigma, & \text{if } y \in \overline{\mathrm{St}}\,\sigma \setminus \big(\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big), \\ y, & \text{if } y \in |P| \setminus (\overline{\mathrm{St}}\,\sigma). \end{cases}$$

If $\overline{\mathrm{St}}\,\sigma = \sigma$ define

$$g(y) := g_z(y) := \begin{cases} s(y,0), & \text{if } y \in \sigma \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big], \\ y, & \text{if } y \in |P| \setminus \sigma. \end{cases} \tag{2.36}$$

We will see ((A.27)) that g is a continuous map of $|P| \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big]$ into itself. Define

$$\Phi'(y) := \Phi'_z(y) := \Phi \circ g_z(y) \in \mathsf{F}, \quad y \in |P|, \tag{2.37}$$

whenever the right hand side is defined. Define also

$$\mathcal{S}' = \mathcal{S}'_z = g_z^{-1}\big[\mathcal{S} \setminus \sigma\big] \cup \big[\mathcal{C}_z \cap (\mathrm{Bd}\,\sigma)\big]. \tag{2.38}$$

The following gives some properties of Φ' and $\mathcal{S}'$. Its proof, in subsection A in the appendix, develops the properties of $\sigma(\cdot)$ and g.

Lemma 2.1. *Let $\mathcal{S}$ be a nonempty closed subset of $|P|$. Let $\sigma \in Q$ be an n-simplex ($n > 0$), let $\mathcal{A} = \mathcal{S} \cap \sigma$, and suppose* (2.3) *holds. Let $z \in (\mathit{Int}\,\sigma) \setminus \mathcal{S}$. Then Φ' and $\mathcal{S}'$, defined by* (2.37) *and* (2.38) *have the following properties.*

(a) *$\mathcal{S}' \cap \sigma = \mathcal{C}_z \cap (\mathit{Bd}\,\sigma)$. In particular, by* (2.6)*, $\mathcal{S} \cap (\mathit{Bd}\,\sigma) \subset \mathcal{S}' \cap (\mathit{Bd}\,\sigma)$ and $\mathcal{S}' \cap (\mathit{Int}\,\sigma) = \varnothing$. Moreover,*

$$\textit{for every } s \geq 0, \textit{ if } \mathcal{H}^s(\mathcal{A}) = 0 \textit{ then } \mathcal{H}^s\big[\mathcal{C}_z \cap (\mathit{Bd}\,\sigma)\big] = 0.$$

In particular,

$$\dim(\mathcal{S}' \cap \sigma) = \dim\big[\mathcal{C}_z \cap (\mathit{Bd}\,\sigma)\big] \leq \dim \mathcal{A} = \dim(\mathcal{S} \cap \sigma). \tag{2.39}$$

(b) *If $\rho \in P$ is not a face of σ then g is one-to-one on $\mathit{Int}\,\rho$, $g(\mathit{Int}\,\rho) = \mathit{Int}\,\rho = g^{-1}(\mathit{Int}\,\rho)$, and the restriction, $g|_{\mathit{Int}\,\rho}$, has a locally Lipschitz inverse on $\mathit{Int}\,\rho$.*

(c) *$\mathcal{S}'$ is closed and Φ' is defined and continuous atE:z.notin.A every point of $|P| \setminus \mathcal{S}'$. If F is a metric space and Φ is locally Lipschitz on $|P| \setminus \mathcal{S}$ then Φ' is locally Lipschitz on $|P| \setminus \mathcal{S}'$.*

(d) *If $\rho \in P$ and $\mathit{Int}\,\rho \subset \mathcal{S}$ then $\mathit{Int}\,\rho \subset \mathcal{S}'$.*

(e) *If $\tau \in P$ and $\tau \cap \sigma = \varnothing$ then $\mathcal{S}' \cap \tau = \mathcal{S} \cap \tau$ and if $y \in \tau \setminus \mathcal{S}$, then $\Phi'(y) = \Phi(y)$. In particular, $\mathcal{S}' \cap (Lk\,\sigma) = \mathcal{S} \cap (Lk\,\sigma)$.*

(f) *If $\tau \neq \sigma$ is a simplex in P of dimension no greater than $n := \dim \sigma$, then $\mathcal{S}' \cap (\tau \setminus \sigma) = \mathcal{S} \cap (\tau \setminus \sigma)$ and $\Phi' = \Phi$ on $\tau \setminus (\sigma \cup \mathcal{S})$.*

(g) *Suppose $\tau \in P$ and $\dim \tau < n := \dim \sigma$. Then*

$$(Int\,\tau) \cap \mathcal{S}' = (Int\,\tau) \cap \Big(\mathcal{S} \cup \big[\mathcal{C}_z \cap (Bd\,\sigma)\big]\Big) = (Int\,\tau) \cap \big[\mathcal{S} \cup \bar{h}_{z,\sigma}(\mathcal{S} \cap \sigma)\big].$$

(h) *If $\rho \in P$ is a simplex of dimension at least n then for every $s \geq 0$, if $\mathcal{H}^s\big[\mathcal{S} \cap (Int\,\rho)\big] = 0$ then $\mathcal{H}^s\big[\mathcal{S}' \cap (Int\,\rho)\big] = 0$. In particular, $\dim\big[\mathcal{S}' \cap (Int\,\rho)\big]) \leq \dim\big[\mathcal{S} \cap (Int\,\rho)\big]$. In particular as well, by* (C.4)*, if $(Int\,\rho) \cap \mathcal{S} = \varnothing$, then $(Int\,\rho) \cap \mathcal{S}' = \varnothing$.*

(i) *If $s \geq 0$, $\tau \in P$, and $\mathcal{H}^s\big[\mathcal{S}' \cap (Int\,\tau)\big] > 0$, then either $\mathcal{H}^s\big[\mathcal{S} \cap (Int\,\tau)\big] > 0$ or τ is a proper face of σ and $\mathcal{H}^s\big[\mathcal{S} \cap (Int\,\sigma)\big] > 0$. Taking $s = 0$, we conclude: If $\tau \in P$ and $(Int\,\tau) \cap \mathcal{S}' \neq \varnothing$ then either $(Int\,\tau) \cap \mathcal{S} \neq \varnothing$ or τ is a proper face of σ.*

(j) *We have*

$$\dim(\mathcal{S}' \cap |Q|) \leq \dim(\mathcal{S} \cap |Q|) \text{ and } \dim \mathcal{S}' \leq \dim \mathcal{S}.$$

(k) *If $\rho \in P$ then*

$$\Phi'(\rho \setminus \mathcal{S}') \subset \Phi(\rho \setminus \mathcal{S}). \tag{2.40}$$

(l) *If $\overline{St}\,\sigma = \sigma$ then $\mathcal{S}' \setminus \sigma = \mathcal{S} \setminus \sigma$, $|P| \setminus \big[(Int\,\sigma) \cup \mathcal{S}'\big] \subset |P| \setminus \big[(Int\,\sigma) \cup \mathcal{S}\big]$, and $\Phi' = \Phi$ off $(Int\,\sigma) \cup \mathcal{S}'$.*

(Point l of the lemma does not seem to be used anywhere.) By point (a) of the lemma $\mathcal{S}' \cap (\operatorname{Int} \sigma) = \varnothing$. This is the main goal of the pushing out operation. (See figure 1.) Call the operation of replacing Φ by Φ' and $\mathcal{S}$ by $\mathcal{S}'$ "pushing $\mathcal{S}$ out of $\operatorname{Int} \sigma$ (from z)". Note that $\mathcal{S}$ might not only be pushed onto $\operatorname{Int} \tau$ for $(n-1)$-faces, τ, of σ, but also possibly onto faces of σ of any dimension $< n$.

2.2. **Constructing $\tilde{\mathcal{S}}$ and $\tilde{\Phi}$.** Say that a simplex $\sigma \in P$ is a "partial simplex of $\mathcal{S}$" (or just "partial simplex" if $\mathcal{S}$ is understood) if $(\operatorname{Int} \sigma) \cap \mathcal{S} \neq \varnothing$ and $(\operatorname{Int} \sigma) \setminus \mathcal{S} \neq \varnothing$. Thus, it is for partial simplices that the pushing out operation was defined. (See (2.1).) Note that if σ is a partial simplex of $\mathcal{S}$ and σ is a proper face of a simplex ρ, then, since $\mathcal{S}$ is closed, either $(\operatorname{Int} \rho) \cap \mathcal{S} = \varnothing$ or ρ is also a partial simplex of $\mathcal{S}$.

Let $q = \dim Q$ and, for $j = 1, \ldots, q$, let M_j be the number of j-dimensional simplices in Q and let $\mathbf{M}$ be the set of all q-tuples $(m_q, \ldots, m_1)$ with $0 \leq m_j \leq M_j$ $(j = 1, \ldots, q)$. Order $\mathbf{M}$ by lexicographic ordering. I.e., define $(m_{q1}, \ldots, m_{11}) >_{\mathbf{M}} (m_{q2}, \ldots, m_{12})$ if and only if the following holds. $(m_{q1}, \ldots, m_{11}) \neq (m_{q2}, \ldots, m_{12})$ and if $j = i$ is the largest $i = 1, \ldots, q$ s.t. $m_{i1} \neq m_{i2}$ then $m_{j1} > m_{j2}$. The relation $>_{\mathbf{M}}$ total orders $\mathbf{M}$.

Describe $\mathcal{S}$'s by elements of $\mathbf{M}$ as follows. Note that there is no such thing as a partial 0-dimensional simplex. Let $\boldsymbol{\mu}(\mathcal{S}) = \big(m_q(\mathcal{S}), \ldots, m_1(\mathcal{S})\big) \in \mathbf{M}$, where, for $j = 1, \ldots, q$, $m_j(\mathcal{S})$ is the number of j-dimensional partial simplices of $\mathcal{S}$ in Q. Note that $\boldsymbol{\mu}$ maps the collection of compact subsets of $|P|$ onto $\mathbf{M}$. To see this, let $B \subset Q$ (but B does not have to be a subcomplex). For every $\sigma \in B$, pick a point $x_\sigma \in \operatorname{Int} \sigma$. Then $\mathcal{S} = \{x_\sigma \in |Q| : \sigma \in B\}$ is a closed set whose set of partial simplices is B. So $\boldsymbol{\mu}$ is surjective, but it is not injective. E.g.,

$\boldsymbol{\mu}$ maps both $|P|$ and $\varnothing$ to $(0,\ldots,0)$. Since $>_{\mathbf{M}}$ is a total ordering of $\mathbf{M}$, each $\mathbf{m} \in \mathbf{M}$ has a unique rank. Thus, $(0,\ldots,0)$ is the unique element of $\mathbf{M}$ with rank 1, etc. If $\mathcal{S} \subset |P|$ is closed, let $\mathbf{rank}(\mathcal{S})$ be the rank of $\boldsymbol{\mu}(\mathcal{S})$.

If there are no partial simplices of $\mathcal{S}$ in Q we are done: Just take $\tilde{\mathcal{S}} = \mathcal{S}$ and $\tilde{\Phi} = \Phi$. So assume there is at least one partial simplex in Q. Thus, $\mathbf{rank}(\mathcal{S}) > 1$. Since Q is a finite complex, there is at least one partial simplex in Q of maximal dimension. If $j = 1,\ldots,q$ is that maximal dimension, then $m_q(\mathcal{S}) = \cdots m_{j+1}(\mathcal{S}) = 0$. Suppose $\sigma \in Q$ is a j-dimensional partial simplex and let $\tau \in Q$ have dimension at least j and *not* be a partial simplex of $\mathcal{S}$. By lemma 2.1(a,d,f,h), the simplex σ will not be a partial simplex of $\mathcal{S}'$, but neither will τ. I.e., pushing $\mathcal{S}$ out of a partial simplex in Q of highest dimension among the partial simplices of $\mathcal{S}$ in Q always results in a $\mathcal{S}' \subset |P|$ s.t.

$$\mathbf{rank}(\mathcal{S}') < \mathbf{rank}(\mathcal{S}). \tag{2.41}$$

Let $\mathcal{S} \subset |P|$ be a closed set and $\Phi : |P| \setminus \mathcal{S} \to \mathsf{F}$ (F an arbitrary topological space) be continuous. Suppose $\dim(\mathcal{S} \cap |Q|) \leq a$, where $a \geq 0$. The set $\tilde{\mathcal{S}}$ posited in theorem 1.1 is obtained from $\mathcal{S}$ as follows. If $\mathcal{S}$ has no partial simplices in Q then take $\tilde{\mathcal{S}} = \mathcal{S}$. Otherwise, begin with a partial simplex, $\sigma \in Q$, of $\mathcal{S}$ of maximal dimension and push $\mathcal{S}$ out of it from an appropriate $z \in \operatorname{Int}\sigma$. (Defining "appropriate" and showing that an appropriate z exists is the business of subsection 2.4.) Then by lemma 2.1(c) we obtain a closed subset, $\mathcal{S}'$ of $|P|$. Moreover, also by point (c) of the lemma, the new map Φ' is continuous off $\mathcal{S}'$ and, by point (j) of lemma 2.1, $\dim(\mathcal{S}' \cap |Q|) \leq \dim(\mathcal{S} \cap |Q|) \leq a$. Now replace $\mathcal{S}$ by $\mathcal{S}'$ and Φ by Φ' and repeat: Find a partial simplex $\sigma \in Q$ of $\mathcal{S}$ of maximal dimension, etc. By (2.41) this procedure terminates after a finite number of steps resulting in a pair $(\tilde{\mathcal{S}}, \tilde{\Phi})$ s.t. $\tilde{\mathcal{S}}$ has no partial simplices. Thus, $\mathbf{rank}(\tilde{\mathcal{S}}) = 1$. Hence, $\tilde{\mathcal{S}} \cap |Q|$ is either empty or is a subcomplex of $|Q|$.

We prove all but parts (9 and 10) of the theorem by induction on $\mathbf{rank}(\mathcal{S})$. If $\mathbf{rank}(\mathcal{S}) = 1$, i.e., there are no partial simplices of $\mathcal{S}$ in Q then the theorem holds with $\tilde{\mathcal{S}} = \mathcal{S}$ and $\tilde{\Phi} = \Phi$. Let $r \geq 1$ and assume parts (1 through 8) of the theorem hold whenever $\mathbf{rank}(\mathcal{S}) \leq r$. Suppose $\mathbf{rank}(\mathcal{S}) = r + 1$. Choose a partial simplex, $\sigma \in Q$, of $\mathcal{S}$ having maximal dimension among all partial simplices in Q and push $\mathcal{S}$ out of σ to obtain a new pair $(\mathcal{S}', \Phi')$ as in (2.37) and (2.38). By points (c and j) of lemma 2.1, $\mathcal{S} = \mathcal{S}'$ and $\Phi = \Phi'$ satisfy the hypotheses of theorem 1.1. But by (2.41), parts (1 through 8) of the theorem hold for $\mathcal{S} = \mathcal{S}'$ and $\Phi = \Phi'$. Let $\tilde{\mathcal{S}}$ and $\tilde{\Phi}$ in the theorem be the corresponding set and map.

By point (c) of lemma 2.1, Φ' is locally Lipschitz off $\mathcal{S}'$ if Φ is locally Lipschitz off $\mathcal{S}$. Part (1) of the theorem follows by induction.

Then part (2) of the theorem is immediate from point (j) of lemma 2.1 and the induction hypothesis. Part (3) is immediate from the induction hypothesis.

Next, we prove part (4) of theorem 1.1. Let $\sigma \in Q$ be a partial simplex of $\mathcal{S}$ of maximal dimension in Q that will be the simplex in Q from which $\mathcal{S}$ will be pushed to produce $(\mathcal{S}', \Phi')$. First, we prove that part (4) holds with $(\mathcal{S}', \Phi')$ in place of $(\tilde{\mathcal{S}}, \tilde{\Phi})$. Suppose $\tau \in P$ is s.t. if $\rho \in Q$ and $(\operatorname{Int}\rho) \cap \mathcal{S} \neq \varnothing$ then $\tau \cap \rho = \varnothing$. Now $(\operatorname{Int}\sigma) \cap \mathcal{S} \neq \varnothing$ because σ is a partial simplex of $\mathcal{S}$. Therefore, in particular, $\tau \cap \sigma = \varnothing$. So by point (e) of lemma 2.1, part (4) holds with $\tilde{\Phi} = \Phi'$ and $\tilde{\mathcal{S}} = \mathcal{S}'$.

Next, observe that if $\rho \in Q$ and $(\operatorname{Int}\rho) \cap \mathcal{S}' \neq \varnothing$ then $\tau \cap \rho = \varnothing$. For by point (i) of the lemma, either $(\operatorname{Int}\rho) \cap \mathcal{S} \neq \varnothing$ or ρ is a proper face of σ. In the former case, $\tau \cap \rho = \varnothing$ by

assumption on τ. In the latter case, if $\tau \cap \rho \neq \varnothing$ then $\tau \cap \sigma \neq \varnothing$, which again contradicts the assumption on τ. Part (4) of the theorem now follows by induction.

Let $\rho \in P \setminus Q$, let $s \geq 0$, and suppose $\mathcal{H}^s\big[\mathcal{S} \cap (\operatorname{Int} \rho)\big] = 0$. Note that, by (B.6), $(\operatorname{Int} \rho) \cap |Q| = \varnothing$. Hence, by points (f and h) of lemma 2.1 (and (B.6) again), $\mathcal{H}^s\big[\mathcal{S}' \cap (\operatorname{Int} \rho)\big] = 0$. Part (5) of theorem 1.1 is immediate from the induction hypothesis.

Notice that if $\dim \tau = a$, then $\mathcal{H}^{\lfloor a \rfloor}(\mathcal{S}' \cap \tau) = \mathcal{H}^{\lfloor a \rfloor}\big[\mathcal{S}' \cap (\operatorname{Int} \tau)\big]$. Part (6) of the theorem is immediate from point (i) of the lemma, the induction hypothesis, and the fact that a face of a face of a simplex τ is a face of τ itself. Similarly, part (7) of the theorem is immediate from point (i) of lemma 2.1.

By the induction hypothesis, part (8) of the theorem, and point (k) of lemma 2.1, if $\rho \in P$, we have

$$\tilde{\Phi}(\sigma \setminus \tilde{\mathcal{S}}) \subset \Phi(\sigma \setminus \mathcal{S}') \subset \Phi(\rho \setminus \mathcal{S}).$$

This proves part (8) of the theorem.

In summary of the preceding discussion, parts (1 through 8) of the theorem are proved.

2.3. **Magnification in one simplex.** The remainder of the proof is taken up with proving parts (9 and 10) of the theorem. Let $q = \dim Q$. If $q < a$ then $\mathcal{H}^a(|Q|) = 0$, so (1.1) holds with any nonnegative K. So suppose $0 \leq a \leq q$. By assumption, $\dim(\mathcal{S} \cap |Q|) \leq a$. Suppose $a = 0$. Then if $\mathcal{H}^a(\mathcal{S} \cap |Q|) = 0$, then $\mathcal{S} \cap |Q|$ is empty, by (C.4), and we are done: By subsection 2.2, $\tilde{\Phi} = \Phi$ and $\tilde{\mathcal{S}} = \mathcal{S}$.

Assume then that $\mathcal{S} \cap |Q| \neq \varnothing$. With $a = 0$ we have $\dim(\mathcal{S} \cap |Q|) = 0$. Let $\sigma \in P$ be a partial simplex of $\mathcal{S}$ of maximal dimension, n, say. By lemma 2.1(a), (2.11), lemma A.2 (part 7), and lemma C.2, we have $\mathcal{H}^0(\mathcal{S}' \cap \sigma) \leq \mathcal{H}^0(\mathcal{S}' \cap \sigma)$. If $\tau \in P$ and $\dim \tau > n$, then, by maximality of n, $(\operatorname{Int} \tau) \cap \mathcal{S} = \varnothing$. Hence, by lemma 2.1(h), we have $(\operatorname{Int} \tau) \cap \mathcal{S} = \varnothing$. If $\tau \in P$ and $\dim \tau \leq n$, then by lemma 2.1(f), we have $\mathcal{H}^0\big[\mathcal{S}' \cap (\tau \setminus \sigma)\big] = \mathcal{H}^0\big[\mathcal{S} \cap (\tau \setminus \sigma)\big]$. Summing up, we have $\mathcal{H}^0(\mathcal{S}') \leq \mathcal{H}^0(\mathcal{S})$. Therefore, as we recursively construct $\tilde{\mathcal{S}}$ from $\mathcal{S}$ as described in subsection 2.2, at each stage the $\mathcal{H}^a$-measure ($= \mathcal{H}^0$-measure) of $\mathcal{S}'$ is not increased, so (1.1) holds with any $K \geq 1$. So assume

$$a > 0. \tag{2.42}$$

Suppose $a = q = \dim Q$. First, suppose that some q-simplex of Q lies in $\mathcal{S}$. Then (1.1) holds with

$$K = K_1 = \frac{\mathcal{H}^q(|Q|)}{\min\big\{\mathcal{H}^q(\omega) : \omega \text{ is a } q\text{-simplex of } Q\big\}} \geq 1. \tag{2.43}$$

Suppose no q-simplex of Q lies in $\mathcal{S}$. Then when we recursively push $\mathcal{S}$ out of all its partial simplices as described in subsection 2.2, the intersection of the resulting $\tilde{\mathcal{S}}$ with $|Q|$ will lie in $|Q^{(a-1)}|$. Then $\mathcal{H}^q(\tilde{\mathcal{S}} \cap |Q|) = \mathcal{H}^q(\tilde{\mathcal{S}} \cap |Q^{(a-1)}|) = 0$ so we can still use any nonnegative K. Thus, we may assume

$$0 < a < q. \tag{2.44}$$

We may assume a is an integer: Recall that $\lfloor a \rfloor$ is the largest integer less than or equal to a. Suppose $\lfloor a \rfloor < a$. Then if $\sigma \in Q$ and $\dim \sigma > \lfloor a \rfloor$ then either $\mathcal{S} \cap \sigma = \varnothing$ or σ is a partial simplex of $\mathcal{S}$. Hence, by recursively applying the pushing out operation we eventually get $\tilde{\mathcal{S}}$ in the $\lfloor a \rfloor$-skeleton of P. But the $\mathcal{H}^a$ measure of this skeleton is 0 so (1.1) holds for any $K \geq 0$. So assume a is an integer.

If $\mathcal{H}^a(\mathcal{S} \cap |Q|) = \infty$ then (1.1) holds for any $K > 0$. So assume $\mathcal{H}^a(\mathcal{S} \cap |Q|) < \infty$. Recall (see (2.2)) $\mathcal{A} = \mathcal{S} \cap \sigma$. Since $\mathcal{H}^a(\mathcal{S} \cap |Q|) < \infty$ we have

$$\mathcal{H}^a(\mathcal{A}) < \infty. \tag{2.45}$$

Claim: If $\mathcal{H}^a\big[\mathcal{S} \cap (\operatorname{Int}\sigma)\big] = 0$, then $\mathcal{H}^a\big(\mathcal{S}' \cap |Q|\big) \leq \mathcal{H}^a\big(\mathcal{S} \cap |Q|\big)$. To see this, assume $\mathcal{H}^a\big[\mathcal{S} \cap (\operatorname{Int}\sigma)\big] = 0$ and note that, by points (d,h) of lemma 2.1 and the fact that σ has maximal dimension (n) among all partial simplices, we have

$$\mathcal{H}^a\Big(\big[\mathcal{S}' \cap |Q|\big] \setminus |Q^{(n)}|\Big) = \mathcal{H}^a\Big(\big[\mathcal{S} \cap |Q|\big] \setminus |Q^{(n)}|\Big).$$

Hence, it suffices to show $\mathcal{H}^a\big(\mathcal{S}' \cap |Q^{(n)}|\big) \leq \mathcal{H}^a\big(\mathcal{S}' \cap |Q^{(n)}|\big)$. But point (f) of lemma 2.1 then implies that it suffices to show $\mathcal{H}^a(\mathcal{S}' \cap \sigma) \leq \mathcal{H}^a(\mathcal{S} \cap \sigma)$. By point (a) of lemma 2.1, (2.11), (2.9), lemma A.2 part (7), and lemma C.2 we have

$$\begin{aligned}
\mathcal{H}^a(\mathcal{S}' \cap \sigma) &\leq \mathcal{H}^a\big[\mathcal{S} \cap (\operatorname{Bd}\sigma)\big] + \mathcal{H}^a\Big(\bar{h}\big[\mathcal{S} \cap (\operatorname{Int}\sigma)\big]\Big) \\
&= \mathcal{H}^a\big[\mathcal{S} \cap (\operatorname{Bd}\sigma)\big] \\
&\leq \mathcal{H}^a(\mathcal{S} \cap \sigma).
\end{aligned}$$

The claim follows. Therefore, assume

$$\mathcal{H}^a\big[\mathcal{S} \cap (\operatorname{Int}\sigma)\big] > 0; \text{ in particular, } n \geq a. \tag{2.46}$$

In our construction of $\tilde{\Phi}$, $\tilde{\mathcal{S}}$ in subsection 2.2, we only applied the pushing operation in partial simplices of maximal dimension. Assume, therefore, that $n = \dim\sigma$ is no smaller than the dimension of any partial simplex of $\mathcal{S}$ in Q. Let $x \in |Q| \setminus \sigma$. Then by (B.7), there exists $\rho \in Q$ s.t. $x \in \operatorname{Int}\rho$. If $\dim\rho > n$, then, by maximality of n, either $(\operatorname{Int}\rho) \cap \mathcal{S} = \varnothing$ or $(\operatorname{Int}\rho) \cap \mathcal{S} = \operatorname{Int}\rho$. Since $x \notin \sigma$, ρ is not a face of σ. Hence, by (B.6), $\operatorname{Int}\rho \subset \rho \setminus \sigma$. But by lemma 2.1(d,f,h), we have

$$\text{If } \rho \in Q \setminus \{\sigma\} \text{ is not a face of } \sigma \text{ then } (\operatorname{Int}\rho) \cap \mathcal{S}' = (\operatorname{Int}\rho) \cap \mathcal{S}.$$

Therefore, $(\mathcal{S}' \setminus \sigma) \cap |Q| = (\mathcal{S} \setminus \sigma) \cap |Q|$. So to compare $\mathcal{H}^a\big(\mathcal{S}' \cap |Q|\big)$ to $\mathcal{H}^a\big(\mathcal{S} \cap |Q|\big)$ it suffices to compare $\mathcal{H}^a\big(\mathcal{S}' \cap \sigma\big)$ to $\mathcal{H}^a\big(\mathcal{S} \cap \sigma\big)$. But $\mathcal{H}^a\big[\mathcal{S}' \cap (\operatorname{Int}\sigma)\big] = 0$ by lemma 2.1(a). So we only need consider the impact that pushing $\mathcal{S}$ out of σ has on $\operatorname{Bd}\sigma$.

If $n = a$ then for any $z_0 \in \operatorname{Int}\sigma$ we have

$$\mathcal{S}' \cap (\operatorname{Bd}\sigma) \subset |Q|^{(a-1)}$$

so $\mathcal{H}^a\big[\mathcal{S}' \cap (\operatorname{Bd}\sigma)\big] = 0$. Hence, after pushing, $\mathcal{H}^a\big(\mathcal{S}' \cap |Q|\big) \leq \mathcal{H}^a\big(\mathcal{S} \cap |Q|\big)$. So assume $n > a$. Thus, by (2.42)

$$a \text{ is an integer and } q \geq n > a > 0. \tag{2.47}$$

We will find a number $\phi < \infty$, that depends only on a and Q but not on $\mathcal{S}$ s.t. we can always find a $z_0 = z_0(\sigma, \mathcal{S}) \in \operatorname{Int}\sigma$ satisfying

$$\begin{aligned}
\mathcal{H}^a\Big(\bar{h}_{z_0,\sigma}\big[\mathcal{A} \cap (\operatorname{Int}\sigma)\big] \cap \tau\Big) &= \mathcal{H}^a\Big(h\big[\mathcal{A} \cap \zeta(z;\tau) \cap (\operatorname{Int}\sigma); z, \tau\big]\Big) \\
&\leq \phi\, \mathcal{H}^a\big[\mathcal{A} \cap (\operatorname{Int}\sigma)\big] \text{ for every } (n-1)\text{-face, } \tau, \text{ of } \sigma.
\end{aligned} \tag{2.48}$$

(See (2.8), (2.15), (2.17), and (2.19).) Observe that (2.48) implies that

$$\mathcal{H}^a\Big(\bar{h}_{z_0,\sigma}\big[\mathcal{A} \cap (\operatorname{Int}\sigma)\big] \cap \tau\Big) \leq \phi\, \mathcal{H}^a\big[\mathcal{A} \cap (\operatorname{Int}\sigma)\big]$$

holds for any proper face, τ, of σ. However, since every proper face of σ lies in an $(n-1)$-face, we need only consider $(n-1)$-faces τ.

First, we bound above $\mathcal{H}^a\Big(h\big[\mathcal{A}\cap\zeta(z;\tau); z, \tau\big]\Big)$ for $z \in \operatorname{Int}\sigma$. Since by (2.29) $h_{z,\tau}$ is Lipschitz on $\mathcal{A}\cap\zeta(z)$, by lemma C.2, we have $\mathcal{H}^a\Big(h\big[\mathcal{A}\cap\zeta(z;\tau)\cap(\operatorname{Int}\sigma);\ z,\tau\big]\Big) = 0$ if $\mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Int}\sigma)\big] = 0$. Thus, by (2.21)

$$\mathcal{H}^a\Big(h\big[\mathcal{A}\cap\zeta(z;\tau);\ z,\tau\big]\Big) = \mathcal{H}^a(\mathcal{A}), \quad \text{if } \mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Int}\sigma)\big] = 0. \tag{2.49}$$

I.e., if $\mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Int}\sigma)\big] = 0$ then (2.48) holds for any $\phi > 0$. So assume $\mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Int}\sigma)\big] > 0$. In summary, by (2.45), we may assume

$$0 < \mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Int}\sigma)\big] < \infty. \tag{2.50}$$

(See (2.46).)

We apply lemma C.3 to $h = h_{z,\tau}$. Note that $n > n-1 \geq a > 0$ by (2.47). (In particular, $n \geq 2$.) If $y = (y_1, \ldots, y_n)$, write $y^n = (y_1, \ldots, y_{n-1})$, the $(n-1)$-dimensional row vector obtained from y by dropping the last coordinate. Moreover, by (2.24) if $y \in \zeta(z)$ then $y_n < z_n$. Interpreting $h_{z,\tau}$ as a map into $\mathbb{R}^{n-1}$ (see (2.14)), (2.27) becomes

$$h_{z,\tau}(y) = \frac{z_n}{z_n - y_n}(y^n - z^n) + z^n. \tag{2.51}$$

The formula (2.51) defines a point of $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ for y in the open superset, $\mathcal{U} := \{w \in \operatorname{Int}\sigma : w_n < z_n\}$, of $\zeta(z)\cap(\operatorname{Int}\sigma)$. On $\mathcal{U}$ we have

$$Dh_{z,\tau}(y) = \left(\frac{z_n}{z_n - y_n} I_{n-1}, \quad \frac{z_n}{(z_n - y_n)^2}(y^n - z^n)^T\right)^{(n-1)\times n},$$

where I_m is the $m \times m$ identity matrix $(m = 1, 2, \ldots)$ and "T" indicates matrix transposition. Therefore,

$$\begin{aligned} Dh_{z,\tau}(y)^T Dh_{z,\tau}(y) &= \begin{pmatrix} \frac{z_n^2}{(z_n-y_n)^2} I_{n-1} & \frac{z_n^2}{(z_n-y_n)^3}(y^n - z^n)^T \\ \frac{z_n^2}{(z_n-y_n)^3}(y^n - z^n) & \frac{z_n^2}{(z_n-y_n)^4}|y^n - z^n|^2 \end{pmatrix} \\ &= \frac{z_n^2}{(z_n - y_n)^2}\begin{pmatrix} I_{n-1} & (z_n - y_n)^{-1}(y^n - z^n)^T \\ (z_n - y_n)^{-1}(y^n - z^n) & (z_n - y_n)^{-2}|y^n - z^n|^2 \end{pmatrix}. \end{aligned}$$

The vector $\big(-(y^n - z^n), (z_n - y_n)\big)^T$ is an eigenvector of $Dh_{z,\tau}(y)^T Dh_{z,\tau}(y)$ with eigenvalue 0. The vector $\big(y^n - z^n, (z_n - y_n)^{-1}|y^n - z^n|^2\big)^T \in \mathbb{R}^n$ is also an eigenvector with eigenvalue

$$\lambda(y; z)^2 := \frac{|z - y|^2 z_n^2}{(z_n - y_n)^4}.$$

If $n = 2$ then we have accounted for all eigenvalues of $Dh_{z,\tau}(y)^T Dh_{z,\tau}(y)$. If $n > 2$ and $v \in \mathbb{R}^{n-1}$ is any non-zero row vector in $\mathbb{R}^{n-1}$ that is perpendicular to $y^n - z^n$, then $(v, 0)^T$ is an eigenvector of $Dh(y; z,\tau)^T\, Dh(y; z, \tau)$ with eigenvalue $z_n^2/(z_n - y_n)^2$. Since the space of all $(n-1)$-vectors that are perpendicular to $y^n - z^n$ is $(n-2)$-dimensional, we have again accounted for all eigenvalues of $Dh_{z,\tau}(y)^T Dh_{z,\tau}(y)$.

If $y \in \zeta(z)$, then, by (2.17), $y = bx + (1-b)z$ for some $x \in \tau$, $0 < b \leq 1$. Since $\tau \subset \mathbb{R}^{n-1}$, we have $x_n = 0$. Therefore,

$$z - y = b\,(z - x) = b\,(z^n - x^n, z_n).$$

Hence, $z_n - y_n = bz_n$. Moreover, $|z - x|^2 = |z^n - x^n|^2 + z_n^2 \geq z_n^2$. Thus, whether $n > 2$ or not, if $y \in \zeta(z)$

(2.52)
$$\lambda(y; z)^2 = \frac{|z-y|^2 z_n^2}{(z_n - y_n)^4} = \frac{b^2|z-x|^2 z_n^2}{(z_n - y_n)^2(z_n - y_n)^2} = \frac{b^2|z-x|^2 z_n^2}{b^2 z_n^2 (z_n - y_n)^2} = \frac{|z-x|^2}{(z_n - y_n)^2} \geq \frac{z_n^2}{(z_n - y_n)^2}.$$

Thus, the largest eigenvalue of $Dh(y; z, \tau)^T\, Dh(y; z, \tau)$ for $y \in \zeta(z)$ is $\lambda(y; z)^2$. From (2.52) we also conclude

$$\lambda(y; z) = \frac{|z - x|}{z_n - y_n} \leq \frac{diam(\sigma)}{z_n - y_n}, \qquad y \in \zeta(z; \tau). \tag{2.53}$$

Recall that the domain of $h_{z,\tau} = h(\cdot; z, \tau)$ is $\zeta(z; \tau)$. (See (2.15) and (2.17).) Therefore, by (2.50), lemma C.3, and (2.53)

$$\mathcal{H}^a\Big(h\big[\mathcal{A} \cap (\operatorname{Int}\sigma) \cap \zeta(z;\tau); z, \tau\big]\Big) \leq \int_{\mathcal{A}\cap(\operatorname{Int}\sigma)\cap\zeta(z;\tau)} \left(\frac{diam(\sigma)}{z_n - y_n}\right)^a \mathcal{H}^a(dy), \quad z \in \operatorname{Int}\sigma. \tag{2.54}$$

(The integral in the preceding and other integrals we will encounter in subsection 2.4 look potential theoretic (Falconer [Fal90, section 4.3] and Hayman and Kennedy [HK76, Section 5.4.1, pp. 225–229]), but we only need elementary methods.)

2.4. **Bound on average magnification factor and existence of good point from which to push.** Let $v(0), \ldots, v(n) \in \mathbb{R}^n$ be the vertices of σ. By (B.3)

$$\hat{\sigma} = \frac{1}{n+1} \sum_{j=0}^{n} v(j) \tag{2.55}$$

is the barycenter of σ. By (2.47) we have $n + 1 \geq 3$. Let

$$\gamma \in \left(\tfrac{1}{2(n+1)}, 1\right) \tag{2.56}$$

be a constant (to be determined later) and define a simplex lying inside and "concentric" with σ as follows.

$$\sigma_\gamma = \big\{\gamma x + (1-\gamma)\hat{\sigma} : x \in \sigma\big\} \subset \operatorname{Int}\sigma, \tag{2.57}$$

by (B.2). The vertices of σ_γ are just $\gamma v(j) + (1-\gamma)\hat{\sigma}$ ($j = 0, \ldots, n$). If $y = \sum_{j=0}^n \beta_j(y) v(j) \in \sigma_\gamma$, where the $\beta_j(y)$'s are nonnegative and sum to 1, then for some nonnegative $\hat{\beta}_0, \ldots, \hat{\beta}_n$ summing to 1 we have

$$\beta_j = \gamma\hat{\beta}_j + \frac{1-\gamma}{n+1}, \quad (j = 0, \ldots, n).$$

Therefore,

$$\frac{1-\gamma}{n+1} \leq \beta_j(y) \leq \gamma + \frac{1-\gamma}{n+1}, \quad j = 0, \ldots, n. \quad \text{(For each } j \text{ these inequalities are tight.)} \tag{2.58}$$

We compute the average, over $z \in \sigma_\gamma$, of the right hand side (RHS) in (2.54). Let $\mathcal{L}^k$ denote k-dimensional Lebesgue measure ($k = 1, 2, \ldots$). In the following calculation we employ the product measure theorem and Fubini's theorem (Ash [Ash72, Section 2.6]). This is justified

because we employ either Lebesgue measure or the product measure $\left(\mathcal{H}^a|_{\mathcal{A}\cap(\operatorname{Int}\sigma)}\right) \times \mathcal{L}^n$ and the restriction $\mathcal{H}^a|_{\mathcal{A}\cap(\operatorname{Int}\sigma)}$ is a finite measure by (2.45).

First, we show that certain subsets of $\mathbb{R}^{2n}$ are Borel measurable. Recall that if $z \in \mathbb{R}^n$ then we write $z = (z^n, z_n)$ with $z^n \in \mathbb{R}^{n-1}$ and $z_n \in \mathbb{R}$. Let $G : (\operatorname{Int}\sigma) \times \sigma \to \sigma \times \mathbb{R} \times \mathbb{R}^n \times \sigma$ be defined by

$$G(z, y) = G(z^n, z_n, y) = \left((z^n, z_n), \frac{z_n - y_n}{z_n}, \frac{z_n}{z_n - y_n} \left(y - \frac{y_n}{z_n}(z^n, z_n) \right), y \right). \tag{2.59}$$

(Recall that, by (2.24), $z \in \operatorname{Int}\sigma$ implies $z_n > 0$. Define $\frac{z_n}{z_n - y_n}\left(y - \frac{y_n}{z_n} z\right) = 0 \in \mathbb{R}^n$ if $z_n - y_n = 0$.) Then G is Borel measurable. Hence, the set

$$\left\{(z, y) \in \sigma_\gamma \times ((\operatorname{Int}\sigma) \cap \mathcal{A}) : y \in \zeta(z; \tau)\right\} = G^{-1}\left[\sigma_\gamma \times (0, 1) \times \tau \times \mathcal{A}\right]$$

is Borel measurable. (See (2.24), (2.17), (2.25), (2.27), and (2.28).) If S is a set define the "indicator", 1_S, to be the function

$$1_S(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise.} \end{cases}$$

(An indicator function is often called a "characteristic function".) Thus,

$$\text{the indicator function } 1_{\left\{(z,y)\in\sigma_\gamma\times((\operatorname{Int}\sigma)\cap\mathcal{A}):y\in\zeta(z;\tau)\right\}} \text{ is Borel measurable.} \tag{2.60}$$

Similarly, if $y \in \sigma$ and $z_n > 0$, the set

$$Z(y, z_n) := \left\{z^n \in \mathbb{R}^{n-1} : (z^n, z_n) \in \sigma_\gamma \text{ and } y \in \zeta(z^n, z_n)\right\} \tag{2.61}$$

is just the z_n-section (followed by projection onto the first factor) of the Borel set

$$G^{-1}\left[\sigma_\gamma \times (0, 1] \times \tau \times \{y\}\right].$$

(For some choices of y, z_n we have $Z(y, z_n) = \varnothing$.) Therefore, for $y \in \operatorname{Int}\sigma$ fixed, $\mathcal{L}^{n-1}\left[Z(y, z_n)\right]$ is Borel measurable in $z_n \in \mathbb{R}$.

Let $z \in \operatorname{Int}\sigma$. Recall that $v(n)$ is the vertex of σ opposite τ. Let

$$z_{n,max} = z_{n,max}(\sigma, \tau) = \max\{z_n \geq 0 : z \in \sigma\}. \tag{2.62}$$

(See figure 1.) This is just the n^{th} coordinate of $v(n)$. I.e., $z_{n,max} = v_n(n)$. Since $\tau \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n$, the n^{th} coordinates of $v(0), \ldots, v(n-1)$ are 0. Therefore, $z_n = \beta_n(z) z_{n,max}$. Let $z \in \sigma_\gamma$. Then multiplying (2.58), with $y = z$ and $j = n$, through by $z_{n,max}$ we get the following tight inequalities.

$$\frac{1-\gamma}{n+1} z_{n,max} \leq z_n \leq \left(\gamma + \frac{1-\gamma}{n+1}\right) z_{n,max} = \frac{n\gamma + 1}{n+1} z_{n,max}, \text{ for every } z \in \sigma_\gamma. \tag{2.63}$$

Recalling (2.50), (2.63), and (2.24), we can bound the integrated right hand side of (2.54) as follows.

$$\begin{aligned}
(2.64)\quad & \int_{\sigma_\gamma} \int_{\mathcal{A}\cap(\mathrm{Int}\,\sigma)\cap\zeta(z;\tau)} \left(\frac{diam(\sigma)}{z_n - y_n}\right)^a \mathcal{H}^a(dy)\,dz \\
&= \int_{\mathcal{A}\cap(\mathrm{Int}\,\sigma)} \int_{\{z\in\sigma_\gamma : y\in\zeta(z)\}} \left(\frac{diam(\sigma)}{z_n - y_n}\right)^a dz\,\mathcal{H}^a(dy) \\
&= diam(\sigma)^a \int_{\mathcal{A}\cap(\mathrm{Int}\,\sigma)} \int_{\{z\in\sigma_\gamma : y\in\zeta(z)\}} (z_n - y_n)^{-a}\,dz\,\mathcal{H}^a(dy) \\
&= diam(\sigma)^a \int_{\mathcal{A}\cap(\mathrm{Int}\,\sigma)} \int_{\max\left\{\frac{1-\gamma}{n+1} z_{n,max},\, y_n\right\}}^{\frac{n\gamma+1}{n+1} z_{n,max}} \int_{Z(y,z_n)} (z_n - y_n)^{-a}\,dz^n\,dz_n\,\mathcal{H}^a(dy) \\
&\leq diam(\sigma)^a \int_{\mathcal{A}\cap(\mathrm{Int}\,\sigma)} \int_{y_n}^{z_{n,max}} \int_{Z(y,z_n)} (z_n - y_n)^{-a}\,dz^n\,dz_n\,\mathcal{H}^a(dy) \\
&= diam(\sigma)^a \int_{\mathcal{A}\cap(\mathrm{Int}\,\sigma)} \int_{y_n}^{z_{n,max}} \mathcal{L}^{n-1}\big[Z(y,z_n)\big](z_n - y_n)^{-a}\,dz_n\,\mathcal{H}^a(dy).
\end{aligned}$$

If $z = (z^n, z_n) \in \mathbb{R}^n$ with $z_n > 0$, let

$$(2.65)\qquad \zeta^*(z) = \left\{ y \in \mathbb{R}^n : 0 \leq y_n < z_n \text{ and } \frac{z_n}{z_n - y_n}\left(y - \frac{y_n}{z_n} z\right) \in \tau \right\}.$$

Notice that definition (2.17) makes sense for any $z = (z^n, z_n) \in \mathbb{R}^n$ with $z_n > 0$. Therefore, by (2.27), (2.28), and (2.24),

$$\zeta(z) = \zeta^*(z) \cap \sigma.$$

Let $y \in \mathrm{Int}\,\sigma$ and $z_n \in (y_n, z_{n,max})$. For z^n to be in $Z(y, z_n)$ two things must happen, *viz.* $y \in \zeta(z^n, z_n)$ and $(z^n, z_n) \in \sigma_\gamma$. Thus, we can bound $\mathcal{L}^{n-1}\big[Z(y, z_n)\big]$ above by the volume of the set $\big\{z^n \in \mathbb{R}^{n-1} : y \in \zeta^*(z^n, z_n)\big\}$. We can also bound it above by the volume of $\big\{z^n \in \mathbb{R}^{n-1} : (z^n, z_n) \in \sigma_\gamma\big\}$. In summary,

$$\begin{aligned}
(2.66)\quad & \mathcal{L}^{n-1}\big[Z(y,z_n)\big] \\
& \leq \min\Big\{ \mathcal{L}^{n-1}\Big(\big\{z^n \in \mathbb{R}^{n-1} : y \in \zeta^*(z^n, z_n)\big\}\Big), \mathcal{L}^{n-1}\Big(\big\{z^n \in \mathbb{R}^{n-1} : (z^n, z_n) \in \sigma_\gamma\big\}\Big) \Big\}.
\end{aligned}$$

From (2.65) and interpreting τ as a subset of $\mathbb{R}^{n-1}$, we see that if $y \in \zeta^*(z^n, z_n) \cap (\mathrm{Int}\,\sigma)$ then

$$z_n > y_n > 0 \text{ and } \frac{z_n}{z_n - y_n}\left(y^n - \frac{y_n}{z_n} z^n\right) \in \tau.$$

I.e.,

$$(2.67)\qquad z_n > y_n > 0 \text{ and } z^n \in \frac{z_n}{y_n} y^n - \frac{z_n - y_n}{y_n}\tau.$$

By (C.6) in appendix C the volume of the set defined by the right end of the above is

$$(2.68)\qquad \mathcal{L}^{n-1}\left(\frac{z_n}{y_n} y^n - \frac{z_n - y_n}{y_n}\tau\right) = \left(\frac{z_n - y_n}{y_n}\right)^{n-1} \mathcal{L}^{n-1}(\tau).$$

Thus, if $y \in \operatorname{Int}\sigma$ and $z_n \in (y_n, z_{n,max})$ we have by (2.66)

$$\mathcal{L}^{n-1}\big[Z(y, z_n)\big] \leq \left(\frac{z_n - y_n}{y_n}\right)^{n-1} \mathcal{L}^{n-1}(\tau). \tag{2.69}$$

By (2.57), the $(n-1)$-dimensional "base" of σ_γ (i.e., the $(n-1)$-face of σ_γ corresponding to $z_n = (1-\gamma)z_{n,max}/(n+1)$) in (2.63) is $\gamma\tau + (1-\gamma)\hat{\sigma}$. Call this face τ_γ. Then, by (C.6) again, $\mathcal{L}^{n-1}(\tau_\gamma) = \gamma^{n-1}\mathcal{L}^{n-1}(\tau)$. If z_n satisfies (2.63) the corresponding cross-section, $\big\{w \in \mathbb{R}^{n-1} : (w, z_n) \in \sigma_\gamma\big\}$, of σ_γ is a convex combination of the base, τ_γ, of σ_γ and the "top" vertex of σ_γ, which is $\gamma v(n) + (1-\gamma)\hat{\sigma}$. The cross section volume is given by

$$\begin{aligned} \mathcal{L}^{n-1}\Big(\big\{w \in \mathbb{R}^{n-1} : (w, z_n) \in \sigma_\gamma\big\}\Big) &= \left[\frac{\frac{n\gamma+1}{n+1} z_{n,max} - z_n}{\left(\gamma + \frac{1-\gamma}{n+1} - \frac{1-\gamma}{n+1}\right) z_{n,max}}\right]^{n-1} \gamma^{n-1}\mathcal{L}^{n-1}(\tau) \\ &= \left(\frac{\frac{n\gamma+1}{n+1} z_{n,max} - z_n}{z_{n,max}}\right)^{n-1} \mathcal{L}^{n-1}(\tau). \end{aligned} \tag{2.70}$$

Thus, if $y \in \operatorname{Int}\sigma$, $y_n < z_n$, and (2.63) holds, we have by (2.66)

$$\mathcal{L}^{n-1}\big[Z(y, z_n)\big] \leq \left(\frac{\frac{n\gamma+1}{n+1} z_{n,max} - z_n}{z_{n,max}}\right)^{n-1} \mathcal{L}^{n-1}(\tau). \tag{2.71}$$

Suppose $z \in \sigma_\gamma$ and $y \in \operatorname{Int}\sigma$ satisfy $z_n > y_n \geq \frac{(1-\gamma)z_{n,max}}{2(n+1)}$. Then by (2.63) and (2.56) we have

$$\begin{aligned} z_n - y_n &\leq \left[\gamma + \frac{1-\gamma}{n+1} - \frac{1-\gamma}{2(n+1)}\right] z_{n,max} \\ &= \left[\gamma + \frac{1-\gamma}{2(n+1)}\right] z_{n,max} \\ &= \gamma\left[1 + \frac{1-\gamma}{\gamma}\frac{1}{2(n+1)}\right] z_{n,max} \\ &< \gamma\left[1 + \frac{1 - \frac{1}{2(n+1)}}{\frac{1}{2(n+1)}}\frac{1}{2(n+1)}\right] z_{n,max} \\ &= \gamma\left(2 - \frac{1}{2(n+1)}\right) z_{n,max} \\ &< 2\gamma\, z_{n,max}. \end{aligned}$$

Therefore, by (2.47) and (2.69), if $z \in \sigma_\gamma$ and $y \in \operatorname{Int}\sigma$ satisfy $z_n > y_n \geq \frac{(1-\gamma)z_{n,max}}{2(n+1)}$, then

$$\begin{aligned}
\mathcal{L}^{n-1}\big[Z(y,z_n)\big](z_n-y_n)^{-a} &\leq \left(\frac{z_n-y_n}{y_n}\right)^{n-1}(z_n-y_n)^{-a}\,\mathcal{L}^{n-1}(\tau)\\
&\leq \left(\frac{z_n-y_n}{\frac{(1-\gamma)z_{n,max}}{2(n+1)}}\right)^{n-1}(z_n-y_n)^{-a}\,\mathcal{L}^{n-1}(\tau)\\
&\leq \frac{(2\gamma z_{n,max})^{n-a-1}}{\left(\frac{(1-\gamma)z_{n,max}}{2(n+1)}\right)^{n-1}}\mathcal{L}^{n-1}(\tau) \qquad (2.72)\\
&= \frac{2^{2n-a-2}(n+1)^{n-1}\gamma^{n-a-1}(z_{n,max})^{-a}}{(1-\gamma)^{n-1}}\,\mathcal{L}^{n-1}(\tau).
\end{aligned}$$

Now suppose $z \in \sigma_\gamma$ and $0 < y_n < \frac{(1-\gamma)z_{n,max}}{2(n+1)} < \frac{(1-\gamma)z_{n,max}}{n+1}$. Then we have by (2.71), (2.47), and (2.63),

$$\begin{aligned}
(2.73)\quad &\mathcal{L}^{n-1}\big[Z(y,z_n)\big](z_n-y_n)^{-a}\\
&\leq \frac{\left(\frac{n\gamma+1}{n+1}z_{n,max}-z_n\right)^{n-1}}{(z_{n,max})^{n-1}}\left[z_n-\frac{(1-\gamma)z_{n,max}}{2(n+1)}\right]^{-a}\mathcal{L}^{n-1}(\tau)\\
&\leq \frac{\left(\gamma+\frac{1-\gamma}{n+1}-\frac{1-\gamma}{n+1}\right)^{n-1}(z_{n,max})^{n-1}}{(z_{n,max})^{n-1}}\left[\frac{1-\gamma}{n+1}-\frac{1-\gamma}{2(n+1)}\right]^{-a}(z_{n,max})^{-a}\mathcal{L}^{n-1}(\tau)\\
&= \frac{2^a(n+1)^a\gamma^{n-1}}{(1-\gamma)^a}(z_{n,max})^{-a}\mathcal{L}^{n-1}(\tau).
\end{aligned}$$

By (2.47) and (2.56), for any $y \in \operatorname{Int}\sigma$ with $y_n < z_n$, the final value in (2.72) is no smaller than that in (2.73). Thus, if (2.63) holds then for every $y \in \operatorname{Int}\sigma$ with $0 < y_n < z_n$ we have the following regardless if y_n is larger or smaller than $\frac{(1-\gamma)z_{n,max}}{2(n+1)}$.

$$(2.74)\qquad \mathcal{L}^{n-1}\big[Z(y,z_n)\big](z_n-y_n)^{-a} \leq \frac{2^{2n-a-2}(n+1)^{n-1}\gamma^{n-a-1}(z_{n,max})^{-a}}{(1-\gamma)^{n-1}}\,\mathcal{L}^{n-1}(\tau).$$

Then from (2.64) and (2.74) we get

$$\begin{aligned}
(2.75)\quad &\int_{\sigma_\gamma}\int_{\mathcal{A}\cap(\operatorname{Int}\sigma)\cap\zeta(z;\tau)}\left(\frac{diam(\sigma)}{z_n-y_n}\right)^a\mathcal{H}^a(dy)dz\\
&\qquad\leq \left(\frac{diam(\sigma)}{z_{n,max}}\right)^a z_{n,max}\frac{2^{2n-a-2}(n+1)^{n-1}\gamma^n}{(1-\gamma)^{n-1}\gamma^{a+1}}\mathcal{L}^{n-1}(\tau)\,\mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Int}\sigma)\big].
\end{aligned}$$

Let $r(\sigma)$ be the radius and $t(\sigma)$ the thickness of σ (appendix B). *Claim:* $r(\sigma) \leq z_{n,max}/(n+1)$. By (2.14), (2.55), and (2.62), $z_{n,max}/(n+1)$ is the perpendicular distance from the barycenter, $\hat{\sigma}$, to $\mathbb{R}^{n-1}$, thought of as the subspace of $\mathbb{R}^n$ containing τ. A perpendicular dropped from $\hat{\sigma}$ to $\mathbb{R}^{n-1}$ intersects $\mathbb{R}^{n-1}$ at a point $\hat{\sigma}_0$, say. If $\hat{\sigma}_0 \in \tau$, then $z_{n,max}/(n+1)$ is the distance from $\hat{\sigma}$ to τ. If $\hat{\sigma} \notin \tau$ then the line perpendicular to $\mathbb{R}^{n-1}$ and joining $\hat{\sigma}$ to $\hat{\sigma}_0$ must intersect

some other face, τ', of σ. The distance from $\hat{\sigma}$ to τ' will be $< z_{n,max}/(n+1)$. That proves the claim.

It follows that

$$\frac{diam(\sigma)}{z_{n,max}} \leq \frac{1}{(n+1)\,t(\sigma)}. \tag{2.76}$$

Combining (2.75) and (2.76) we get

$$\begin{aligned}\int_{\sigma_\gamma} \int_{\mathcal{A}\cap(\mathrm{Int}\,\sigma)\cap\zeta(z;\tau)} \left(\frac{diam(\sigma)}{z_n - y_n}\right)^a \mathcal{H}^a(dy)dz \\ \leq \frac{2^{2n-a-2} z_{n,max}(n+1)^{n-a-1}\gamma^n}{t(\sigma)^a(1-\gamma)^{n-1}\gamma^{a+1}} \mathcal{L}^{n-1}(\tau)\mathcal{H}^a\big[\mathcal{A}\cap(\mathrm{Int}\,\sigma)\big].\end{aligned} \tag{2.77}$$

Let

$$\begin{aligned} X &= X(\tau) = X(\tau,\sigma) \\ &:= \Bigg\{ z \in \sigma_\gamma : \int_{\mathcal{A}\cap(\mathrm{Int}\,\sigma)\cap\zeta(z;\tau)} \left(\frac{diam(\sigma)}{z_n - y_n}\right)^a \mathcal{H}^a(dy) \\ &\qquad \leq \frac{2^{2n-a-2} n(n+1)^{n-a-1}(n+2)}{(1-\gamma)^{n-1}\gamma^{a+1}\,t(\sigma)^a} \mathcal{H}^a\big[\mathcal{A}\cap(\mathrm{Int}\,\sigma)\big] \Bigg\}. \end{aligned}$$

By (2.60) and (2.45), Fubini (Ash [Ash72, Theorem 2.6.4, p. 101]) tells us X is Borel measurable. Suppose

$$\mathcal{L}^n(X) < \frac{n+1}{n+2}\mathcal{L}^n(\sigma_\gamma). \tag{2.78}$$

Now, by (2.63) and (2.70),

$$\begin{aligned}\mathcal{L}^n(\sigma_\gamma) = \int_{\frac{1-\gamma}{n+1}z_{n,max}}^{\frac{n\gamma+1}{n+1}z_{n,max}} \left(\frac{\frac{n\gamma+1}{n+1}z_{n,max} - z_n}{z_{n,max}}\right)^{n-1} \mathcal{L}^{n-1}(\tau)\,dz_n \\ = \gamma^n z_{n,max}\,\mathcal{L}^{n-1}(\tau)/n > 0.\end{aligned} \tag{2.79}$$

Hence,

$$\frac{\gamma^n z_{n,max}\,\mathcal{L}^{n-1}(\tau)}{n\,\mathcal{L}^n(\sigma_\gamma)} = 1. \tag{2.80}$$

Then by (2.77), (2.80), (2.78), (2.50), and the definition of X

$$\begin{aligned}
&\frac{2^{2n-a-2} z_{n,max}(n+1)^{n-a-1}\gamma^n}{(1-\gamma)^{n-1}\gamma^{a+1}\, t(\sigma)^a}\mathcal{L}^{n-1}(\tau)\mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Int}\sigma)\big] \\
&\quad \geq \int_{\sigma_\gamma}\int_{\mathcal{A}\cap(\operatorname{Int}\sigma)\cap\zeta(z;\tau)}\left(\frac{diam(\sigma)}{z_n-y_n}\right)^a \mathcal{H}^a(dy)\,dz \\
&\quad \geq \mathcal{L}^n(\sigma_\gamma\setminus X)\times\frac{2^{2n-a-2}n(n+1)^{n-a-1}(n+2)}{(1-\gamma)^{n-1}\gamma^{a+1}\, t(\sigma)^a}\mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Int}\sigma)\big] \\
&\quad = \mathcal{L}^n(\sigma_\gamma\setminus X)\times(n+2)\frac{2^{2n-a-2}n(n+1)^{n-a-1}}{(1-\gamma)^{n-1}\gamma^{a+1}\, t(\sigma)^a}\mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Int}\sigma)\big] \\
&\quad = \frac{\gamma^n z_{n,max}\,\mathcal{L}^{n-1}(\tau)}{n\,\mathcal{L}^n(\sigma_\gamma)}\times\mathcal{L}^n(\sigma_\gamma\setminus X)\times(n+2)\frac{2^{2n-a-2}n(n+1)^{n-a-1}}{(1-\gamma)^{n-1}\gamma^{a+1}\, t(\sigma)^a}\mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Int}\sigma)\big] \\
&\quad = \mathcal{L}^n(\sigma_\gamma\setminus X)\times(n+2)\mathcal{L}^n(\sigma_\gamma)^{-1}\gamma^n z_{n,max} \\
&\qquad\qquad \frac{2^{2n-a-2}(n+1)^{n-a-1}}{(1-\gamma)^{n-1}\gamma^{a+1}\, t(\sigma)^a}\mathcal{L}^{n-1}(\tau)\mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Int}\sigma)\big] \\
&\quad > \left(1-\frac{n+1}{n+2}\right)\mathcal{L}^n(\sigma_\gamma)\times(n+2)\mathcal{L}^n(\sigma_\gamma)^{-1}\gamma^n z_{n,max} \\
&\qquad\qquad \times\frac{2^{2n-a-2}(n+1)^{n-a-1}}{(1-\gamma)^{n-1}\gamma^{a+1}\, t(\sigma)^a}\mathcal{L}^{n-1}(\tau)\mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Int}\sigma)\big] \\
&\quad = \gamma^n z_{n,max}\frac{2^{2n-a-2}(n+1)^{n-a-1}}{(1-\gamma)^{n-1}\gamma^{a+1}\, t(\sigma)^a}\mathcal{L}^{n-1}(\tau)\mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Int}\sigma)\big].
\end{aligned}$$

So the extremes of the preceding strict inequality are equal. Contradiction. We conclude that (2.78) is false. I.e.,

$$\mathcal{L}^n\big(X(\tau)\big)\geq\frac{n+1}{n+2}\mathcal{L}^n(\sigma_\gamma)>0. \tag{2.81}$$

For $r=a+1,a+2,\ldots$ and $t>0$, let

$$\tilde{\phi}(a,r,t,\gamma)=\frac{2^{2r-a-2}r(r+1)^{r-a-1}(r+2)}{(1-\gamma)^{r-1}\gamma^{a+1}\, t^a}. \tag{2.82}$$

Thus,

$$\begin{aligned}X(\tau)=\Bigg\{z\in\sigma_\gamma: \int_{\mathcal{A}\cap(\operatorname{Int}\sigma)\cap\zeta(z;\tau)}\left(\frac{diam(\sigma)}{z_n-y_n}\right)^a \mathcal{H}^a(dy) \\
\leq\tilde{\phi}(a,n,t(\sigma),\gamma)\mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Int}\sigma)\big]\Bigg\}.\end{aligned} \tag{2.83}$$

Define $g:\gamma\mapsto(1-\gamma)^{r-1}\gamma^{a+1}\quad(0\leq\gamma\leq 1)$. Suppose a and $n=r$ satisfy (2.47). I.e., suppose

$$r\geq a+1 \text{ and } a\geq 1. \tag{2.84}$$

Now, g is nonnegative on $[0,1]$ and $g(0) = g(1) = 0$. Therefore, g must have a local maximum in $(0,1)$. The maximum value of g is achieved at $\gamma = \frac{a+1}{r+a} \in \left(\frac{1}{2(r+1)}, 1\right)$. The maximum value[1] of g is

$$c_{ra} := g\left(\tfrac{a+1}{r+a}\right) = \frac{(r-1)^{r-1}(a+1)^{a+1}}{(r+a)^{r+a}}.$$

Let

$$\phi(a,r,t) = \tilde{\phi}\left(a,r,t,\tfrac{a+1}{r+a}\right). \tag{2.85}$$

So

$$\phi(a,r,t) \leq \tilde{\phi}(a,r,t,\gamma) \text{ for all } \gamma \in \left(\tfrac{1}{2(n+1)}, 1\right). \tag{2.86}$$

Moreover, it is easy to see that c_{ra} is decreasing in r and, hence,

$$\phi(a,r,t) \text{ is increasing in } r > 1. \tag{2.87}$$

Let $q = \dim Q$, let

$$t_{min}(Q) = \min\bigl\{t(\sigma') : \sigma' \text{ is an } m\text{-simplex of } Q \text{ with } m > a\bigr\} \tag{2.88}$$

and let

$$\phi = \phi(Q) = \phi\bigl(a, q, t_{min}(Q)\bigr). \tag{2.89}$$

Note that by (2.82)

$$\begin{aligned}\phi \text{ depends on } a, \text{ it only depends on } \sigma \text{ through } Q,\\ \text{and only depends on } Q \text{ through } q \text{ and } t_{min}(Q).\end{aligned} \tag{2.90}$$

If τ' is an $(n-1)$-face of σ, let

$$Y(\tau') = Y(\tau', \sigma) = \left\{z \in \sigma_\gamma : \int_{\mathcal{A}\cap(\mathrm{Int}\,\sigma)\cap\zeta(z;\tau')} \left(\frac{diam(\sigma)}{z_n - y_n}\right)^a \mathcal{H}^a(dy) \leq \phi\,\mathcal{H}^a\bigl[\mathcal{A} \cap (\mathrm{Int}\,\sigma)\bigr]\right\}.$$

As was the case with the set X, the set $Y(\tau')$ is Borel. Now, by (2.83), (2.86), and (2.87), $X(\tau') \subset Y(\tau')$ so, by (2.81), for every $(n-1)$-face, τ', of σ,

$$\mathcal{L}^n\bigl(Y(\tau')\bigr) \geq \frac{n+1}{n+2}\mathcal{L}^n(\sigma_\gamma) > 0. \tag{2.91}$$

[1] *Claim:* We have $c_{ra} \geq 2^{-a-r}$. The claim is true when $r = 2$ (which means $a = 1$ by (2.84)), the smallest possible value of r. Next, suppose $r \geq 3$ and r and a satisfy (2.84). Take the logarithm of $c_{ra}/2^{-a-r}$ and divide by $a + r$.

$$\frac{r-1}{a+r}\log(r-1) + \frac{a+1}{a+r}\log(a+1) - \log(a+r) + \log 2.$$

Differentiating this expression w.r.t. r we find that it is minimized when $r = a + 2$. The claim is proved.

Let $F_{n-1}(\sigma)$ denote the collection of $(n-1)$-faces of σ. Then $F_{n-1}(\sigma)$ has $n+1$ elements. It follows from (2.91)

$$\begin{aligned}
\mathcal{L}^n\left(\bigcap_{\tau'\in F_{n-1}(\sigma)} Y(\tau')\right) &= \mathcal{L}^n(\sigma_\gamma) - \mathcal{L}^n\left(\bigcup_{\tau'\in F_{n-1}(\sigma)} \big[\sigma_\gamma \setminus Y(\tau')\big]\right)\\
&\geq \mathcal{L}^n(\sigma_\gamma) - \sum_{\tau'\in F_{n-1}(\sigma)} \mathcal{L}^n\big[\sigma_\gamma \setminus Y(\tau')\big]\\
&\geq \mathcal{L}^n(\sigma_\gamma)\left[1-(n+1)\left(1-\frac{n+1}{n+2}\right)\right]\\
&= \frac{1}{n+2}\mathcal{L}^n(\sigma_\gamma) > 0.
\end{aligned} \tag{2.92}$$

In particular

$$\bigcap_{\tau'\in F_{n-1}(\sigma)} Y(\tau') \neq \varnothing.$$

Let

$$z_0 \in \bigcap_{\tau'\in F_{n-1}(\sigma)} Y(\tau'). \tag{2.93}$$

Then by the definition of the $Y(\tau)$'s and (2.54) we have that (2.48) holds. Thus, no matter what $\mathcal{A}$ is, so long as it has properties (2.3) and (2.50), there exists $z_0 \in \operatorname{Int}\sigma$ s.t. the magnification factor of z_0 and $\mathcal{A}$ is not greater than $(n+1)\phi$ and ϕ does not depend on $\mathcal{A}$. (This is what is meant by "appropriate" in subsection 2.2.)

2.5. **Recursion and totaling.** Let $\mathcal{S} \subset |P|$ be closed and suppose $\dim\big(\mathcal{S}\cap|Q|\big) \leq a$. Let $\sigma \in Q$ be a partial simplex of $\mathcal{S}$. Define $\Phi' = \Phi'_z$ ($\mathcal{S}' = \mathcal{S}'_z$) as in (2.37) (resp. (2.38)) with $z = z_0$ as in (2.93). Let $n = \dim\sigma$. Let τ be a proper face of σ. By lemma 2.1 (point a), (2.11), (2.9), and (2.48) and recalling that, by (2.2), $\mathcal{A} = \mathcal{S}\cap\sigma$, we have

$$\begin{aligned}
\mathcal{H}^a\big[\mathcal{S}'\cap(\operatorname{Int}\tau)\big] &= \mathcal{H}^a\big[\mathcal{C}\cap(\operatorname{Bd}\sigma)\cap(\operatorname{Int}\tau)\big]\\
&= \mathcal{H}^a\Big(\big[\bar{h}(\mathcal{A}\cap(\operatorname{Bd}\sigma))\cup\bar{h}(\mathcal{A}\cap(\operatorname{Int}\sigma))\big]\cap(\operatorname{Int}\tau)\Big)\\
&\leq \mathcal{H}^a\Big(\big[\mathcal{A}\cap(\operatorname{Int}\tau)\big]\cup\big[\bar{h}(\mathcal{A}\cap(\operatorname{Int}\sigma))\cap(\operatorname{Int}\tau)\big]\Big)\\
&\leq \mathcal{H}^a(\mathcal{A}\cap(\operatorname{Int}\tau)) + \mathcal{H}^a\big[\bar{h}(\mathcal{A}\cap(\operatorname{Int}\sigma))\cap(\operatorname{Int}\tau)\big]\\
&\leq \mathcal{H}^a(\mathcal{A}\cap(\operatorname{Int}\tau)) + \phi\mathcal{H}^a(\mathcal{A}\cap(\operatorname{Int}\sigma)).
\end{aligned} \tag{2.94}$$

Summing the inequality (2.94) over all $2^{n+1}-2$ (nonempty) proper faces, τ, of σ, applying (B.7), and recalling that, by lemma 2.1(a), $\mathcal{S}'\cap(\operatorname{Int}\sigma) = \varnothing$, we get

$$\mathcal{H}^a(\mathcal{S}'\cap\sigma) \leq \mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Bd}\sigma)\big] + (2^{n+1}-2)\phi\mathcal{H}^a\big[\mathcal{A}\cap(\operatorname{Int}\sigma)\big]. \tag{2.95}$$

Next, we develop two similar approaches to proving part (9) of theorem 1.1. The first approach uses (2.95) to quickly shows the existence of a $K < \infty$ s.t. (1.1) holds. The second approach computes a (probably wildly too big) expression for such a K. The second approach is useful for proving part (10) of the theorem.

2.5.1. *Induction on* **rank**. To prove part (9) of theorem 1.1 we use induction as in subsection 2.2. If $\mathbf{rank}(\mathcal{S}) = 1$, i.e., there are no partial simplices of $\mathcal{S}$ in Q then theorem 1.1 holds with $\tilde{\mathcal{S}} = \mathcal{S}$, $\tilde{\Phi} = \Phi$, and any $K \geq 1$. Let $r \geq 1$ and assume parts (1 through 9) of theorem 1.1 hold whenever $\mathbf{rank}(\mathcal{S}) \leq r$. In particular, whenever $\mathbf{rank}(\mathcal{S}) \leq r$ there is a constant $K_r < \infty$ s.t. (1.1) holds. Suppose $\mathbf{rank}(\mathcal{S}) = r + 1$.

Choose a partial simplex, $\sigma \in Q$, of $\mathcal{S}$ having maximal dimension. Let $n = \dim \sigma$. Then there are no partial simplices of $\mathcal{S}$ in Q of dimension $> n$. Hence, if $\rho \in Q$ has dimension greater than n, then either $\operatorname{Int} \rho = \varnothing$ or $\operatorname{Int} \rho \subset \mathcal{S}$. Push $\mathcal{S}$ out of σ obtaining a new pair $(\mathcal{S}', \Phi')$ as in lemma 2.1 so that (2.95) holds. Then by (2.41) $\mathbf{rank}(\mathcal{S}) \leq r$. Hence, by the induction hypothesis there is a compact subset $\tilde{\mathcal{S}} = \widetilde{(\mathcal{S}')} \subset |P|$ (and a corresponding map $\tilde{\Phi} = \widetilde{(\Phi')}$) satisfying parts (1 through 9) of theorem 1.1 (including (1.1) with $K = K_r < \infty$) with $\mathcal{S}$ and Φ replaced by $\mathcal{S}'$ and Φ', resp. In subsection 2.2 we showed that $(\tilde{\Phi}, \tilde{\mathcal{S}})$ satisfies parts $(1 - 8)$ of theorem 1.1. Moreover, by (B.6), lemma 2.1 (d, f, h), and (2.95), we have

$$
\begin{aligned}
\mathcal{H}^a(\tilde{\mathcal{S}} \cap |Q|) &\leq K_r \mathcal{H}^a(\mathcal{S}' \cap |Q|) \\
&= K_r \mathcal{H}^a\big[(\mathcal{S}' \setminus \sigma) \cap |Q|\big] + K_r \mathcal{H}^a(\mathcal{S}' \cap \sigma) \\
&= K_r \sum_{\rho \in Q,\, \rho \nsubseteq \sigma} \mathcal{H}^a((\operatorname{Int} \rho) \cap \mathcal{S}') + K_r \mathcal{H}^a(\mathcal{S}' \cap \sigma) \\
&= K_r \sum_{\rho \in Q,\, \rho \nsubseteq \sigma} \mathcal{H}^a((\operatorname{Int} \rho) \cap \mathcal{S}) + K_r \mathcal{H}^a(\mathcal{S}' \cap \sigma) \\
&= K_r \mathcal{H}^a\big[(\mathcal{S} \setminus \sigma) \cap |Q|\big] + K_r \mathcal{H}^a(\mathcal{S}' \cap \sigma) \\
&\leq K_r \mathcal{H}^a\big[(\mathcal{S} \setminus \sigma) \cap |Q|\big] + K_r \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Bd} \sigma)) + (2^{n+1} - 2) K_r \phi \mathcal{H}^a\big[\mathcal{S} \cap (\operatorname{Int} \sigma)\big] \\
&\leq (2^{n+1} - 2) K_r (\phi + 1)\, \mathcal{H}^a(\mathcal{S} \cap |Q|).
\end{aligned}
$$

Thus, (1.1) holds for compact $\mathcal{S} \subset |P|$ with $\mathbf{rank}(\mathcal{S}) \leq r + 1$. So now we have proved parts (1 through 9) of the theorem. But to prove part (10) of the theorem we need a more explicit expression for K.

2.5.2. *More explicit expression for K.* Let $q = \dim Q$. Let $k = 0, 1, 2, \ldots, q$. Recall that $Q^{(k)}$ denotes the k-skeleton of Q, i.e., the collection of all simplices in Q of dimension no greater than k. Then $|Q^{(k)}|$ is the polytope, i.e., union of these simplices. Let $(Q^{(k)})^c$ denote the collection of simplices of Q of dimension strictly greater than k. Let $\partial Q^{(k)}$ denote the collection of simplices of Q of dimension exactly k. If τ is a proper face of a simplex σ write $\tau \prec \sigma$ (Munkres [Mun84, p. 86]). If $\tau \prec \sigma$ or $\tau = \sigma$ write $\tau \preccurlyeq \sigma$. We apply the recursive pushing procedure described in section 2.2. Extend the idea of pushing in a trivial way as follows. If $\sigma \in Q$ is not a partial simplex of $\mathcal{S}$, define *pushing $\mathcal{S}$ out of σ* to mean making no change to either $\mathcal{S}$ or Φ. To be more explicit, this convention amounts to this extension of (2.38).

(2.38') $$\mathcal{S}' = \mathcal{S} \text{ if } \sigma \text{ is not a partial simplex of } \mathcal{S}.$$

Moreover, if $\sigma \in Q$ is not a partial simplex of $\mathcal{S}$ and $z \in \operatorname{Int} \sigma$, define $\bar{h} = \bar{h}_{z,\sigma}$ (see (2.8)) to just be the identity on σ:

(2.96) $$\bar{h}(x) := \bar{h}_\sigma(x) := \bar{h}_{z,\sigma}(x) := x, \quad x \in \sigma, \text{ if } \sigma \in Q \text{ is not a partial simplex of } \mathcal{S}.$$

(In particular, in this case $\bar{h}(x) \notin \operatorname{Bd} \sigma$ in general.) In this way we can apply the pushing procedure to all simplices of Q. In addition, with these extensions, lemma 2.1 (points f, g) continue to hold, because if σ is not a partial simplex of $\mathcal{S}$ then $\bar{h}(\mathcal{S} \cap \sigma) = \mathcal{C}_z \cap (\operatorname{Bd} \sigma) \subset \mathcal{S}$. For if $\operatorname{Int} \sigma \subset \mathcal{S}$ then, since $\mathcal{S}$ is closed, $\operatorname{Bd} \sigma \subset \mathcal{S}$. On the other hand, if $(\operatorname{Int} \sigma) \cap \mathcal{S} = \varnothing$, then, by (2.6), (2.11), and(2.9), we have

$$\mathcal{S} \cap (\operatorname{Bd} \sigma) \subset \mathcal{C}_z \cap (\operatorname{Bd} \sigma) = \bar{h}(\mathcal{S} \cap \sigma) = \bar{h}\big[\mathcal{S} \cap (\operatorname{Bd} \sigma)\big] = \mathcal{S} \cap (\operatorname{Bd} \sigma).$$

Let $(\mathcal{S}'_{q,0}, \Phi'_{q,0}) := (\tilde{\mathcal{S}}_q, \tilde{\Phi}_q) := (\mathcal{S}, \Phi)$. Let $a < d \leq q$ and let $\tilde{\mathcal{S}}_d$ be a compact subset of $|P|$ and let $\tilde{\Phi}_d : |P| \setminus \tilde{\mathcal{S}}_d \to \mathsf{F}$ be continuous. The pair $(\tilde{\mathcal{S}}_d, \tilde{\Phi}_d)$ is obtained through repeated pushing. Suppose $\dim\big(\tilde{\mathcal{S}}_d \cap |Q|\big) \leq a$ and no partial simplex of $\tilde{\mathcal{S}}_d$ has dimension higher than d. Then $\tilde{\mathcal{S}}_d \cap |Q| \subset |Q^{(d)}|$. The reason for this is as follows. Let $\sigma \in Q$ and suppose $\dim \sigma > d$. Then by assumption, σ is not a partial simplex of $\tilde{\mathcal{S}}_d$. And we cannot have $\operatorname{Int} \sigma \subset \tilde{\mathcal{S}}_d$ either because otherwise $\dim\big(\tilde{\mathcal{S}}_d \cap |Q|\big) \geq d > a$. Therefore, $\operatorname{Int} \sigma \subset \tilde{\mathcal{S}}_d = \varnothing$. So if $x \in \tilde{\mathcal{S}}_d \cap |Q|$ then, by (B.7), x lies in the interior of some simplex of Q of dimension no greater than d. I.e., $\tilde{\mathcal{S}}_d \cap |Q| \subset |Q^{(d)}|$, as desired.

Write $(\mathcal{S}'_{d,0}, \Phi'_{d,0}) := (\tilde{\mathcal{S}}_d, \tilde{\Phi}_d)$. Push $\mathcal{S}'_{d,0}$ out of some d-simplex in Q. Call the result $(\mathcal{S}'_{d,1}, \Phi'_{d,1})$. If there is another d-simplex in Q then push $\mathcal{S}'_{d,1}$ out of that one, producing $(\mathcal{S}'_{d,2}, \Phi'_{d,2})$. Repeat this process producing a sequence $(\mathcal{S}'_{d,1}, \Phi'_{d,1}), \ldots, (\mathcal{S}'_{d,M_d}, \Phi'_{d,M_d})$, where M_d is the number of d-simplices in Q. By lemma 2.1(j,h), for $m = 0, \ldots, M_d$, we have $\dim\big(\mathcal{S}'_{d,m} \cap |Q|\big) \leq a$ and there are no partial simplices of $\mathcal{S}'_{d,m}$ of dimension greater than d. For the same reason and point (a) of lemma 2.1, there are no partial simplices of $\mathcal{S}'_{d,M_d}$ of dimension greater than $d-1$. Let $(\tilde{\mathcal{S}}_{d-1}, \tilde{\Phi}_{d-1}) := (\mathcal{S}'_{d,M_d}, \Phi'_{d,M_d})$. So as before $\tilde{\mathcal{S}}_{d-1} \cap |Q| \subset |Q^{(d-1)}|$. Let $(\mathcal{S}'_{q,0}, \Phi'_{q,0}) := (\tilde{\mathcal{S}}_q, \tilde{\Phi}_q) := (\mathcal{S}, \Phi)$. Apply this operation recursively and let $(\tilde{\mathcal{S}}, \tilde{\Phi}) := (\tilde{\mathcal{S}}_0, \tilde{\Phi}_0)$.

There are M_q simplices of dimension q in Q. Denote them by $\sigma_1, \ldots, \sigma_{M_q}$, where the ordering is chosen so that $\mathcal{S}'_{q,i-1}$ is pushed out of σ_i which produces $\mathcal{S}'_{q,i}$ ($i = 1, \ldots, M_q$; recall that $\mathcal{S}'_{q,0} = \mathcal{S}$). Let $\bar{h}^i := \bar{h}_{\sigma_i}$ be the map $\bar{h}$ in the simplex σ_i. (See (2.8) and (2.96).) We use the following fact. Let $\tau \in Q^{(q-1)}$ and $r = 1, \ldots, M_q$. Then,

$$\mathcal{S}'_{q,r} \cap (\operatorname{Int} \tau) \subset \big[\mathcal{S} \cap (\operatorname{Int} \tau)\big] \cup \bigcup_{i=1}^{r} \big[\bar{h}^i\big[\mathcal{S} \cap (\operatorname{Int} \sigma_i)\big] \cap (\operatorname{Int} \tau)\big]. \tag{2.97}$$

To prove this, first suppose $r = 1$. Then by lemma 2.1(g) (as extended above and using the modified definition (2.96) of $\bar{h}^1$ if appropriate),

$$\mathcal{S}'_{q,1} \cap (\operatorname{Int} \tau) = (\operatorname{Int} \tau) \cap \big[\mathcal{S} \cup \bar{h}^1(\mathcal{S} \cap \sigma_1)\big],$$

which is (2.97) in the case $r = 1$. Let $m \geq 1$ and suppose (2.97) holds for any $r \leq m$. Now suppose $r = m + 1$. Then by lemma 2.1(g) again and the induction hypothesis,

$$\begin{aligned}\mathcal{S}'_{q,r} \cap (\operatorname{Int}\tau) &= \mathcal{S}'_{q,m+1} \cap (\operatorname{Int}\tau) \\ &= (\operatorname{Int}\tau) \cap \big[\mathcal{S}'_{q,m} \cup \bar{h}^{m+1}(\mathcal{S}'_{q,m} \cap \sigma_{m+1})\big] \\ &= \big[(\operatorname{Int}\tau) \cap \mathcal{S}'_{q,m}\big] \cup \big[(\operatorname{Int}\tau) \cap \bar{h}^{m+1}(\mathcal{S}'_{q,m} \cap \sigma_{m+1})\big] \\ &\subset \left(\big[\mathcal{S} \cap (\operatorname{Int}\tau)\big] \cup \bigcup_{i=1}^{m} \Big[\bar{h}^i\big(\mathcal{S} \cap (\operatorname{Int}\sigma_i)\big) \cap (\operatorname{Int}\tau)\Big]\right) \\ &\qquad \cup \big[(\operatorname{Int}\tau) \cap \bar{h}^{m+1}(\mathcal{S}'_{q,m} \cap \sigma_{m+1})\big].\end{aligned} \tag{2.98}$$

Now,

$$\bar{h}^{m+1}(\mathcal{S}'_{q,m} \cap \sigma_{m+1}) = \bar{h}^{m+1}\big[\mathcal{S}'_{q,m} \cap (\operatorname{Int}\sigma_{m+1})\big] \cup \bar{h}^{m+1}\big[\mathcal{S}'_{q,m} \cap (\operatorname{Bd}\sigma_{m+1})\big]. \tag{2.99}$$

Now by lemma 2.1(f) (as extended above), and noting that if $j < m + 1$, then $\mathcal{S}_{q,j}$ is obtained by pushing $\mathcal{S}'_{q,j-1}$ out of $\sigma_{j-1} \neq \sigma_{m+1}$,

$$\begin{aligned}\mathcal{S}'_{q,m} \cap (\operatorname{Int}\sigma_{m+1}) &= \mathcal{S}'_{q,m-1} \cap (\operatorname{Int}\sigma_{m+1}) \\ &\cdots = \mathcal{S}'_{q,1} \cap (\operatorname{Int}\sigma_{m+1}) = \mathcal{S}'_{q,0} \cap (\operatorname{Int}\sigma_{m+1}) = \mathcal{S} \cap (\operatorname{Int}\sigma_{m+1}).\end{aligned} \tag{2.100}$$

Also, by (2.9)

$$\bar{h}^{m+1}\big[\mathcal{S}'_{q,m} \cap (\operatorname{Bd}\sigma_{m+1})\big] = \mathcal{S}'_{q,m} \cap (\operatorname{Bd}\sigma_{m+1}). \tag{2.101}$$

Substituting (2.100) and (2.101) into (2.99) and recalling that $r = m + 1$, we get

$$\begin{aligned}(\operatorname{Int}\tau)\cap\bar{h}^{m+1}(\mathcal{S}'_{q,m}\cap\sigma_{m+1}) &= \Big(\bar{h}^{m+1}\big[\mathcal{S}\cap(\operatorname{Int}\sigma_{m+1})\big]\cap(\operatorname{Int}\tau)\Big)\cup\Big(\big[\mathcal{S}'_{q,m}\cap(\operatorname{Bd}\sigma_{m+1})\big]\cap(\operatorname{Int}\tau)\Big) \\ &\subset \Big(\bar{h}^r\big[\mathcal{S} \cap (\operatorname{Int}\sigma_r)\big] \cap (\operatorname{Int}\tau)\Big) \cup \Big(\mathcal{S}'_{q,m} \cap (\operatorname{Int}\tau)\Big)\end{aligned} \tag{2.102}$$

Applying the induction hypothesis to $\mathcal{S}'_{q,m} \cap (\operatorname{Int}\tau)$ and then substituting (2.102) into (2.98) proves (2.97).

By (2.11) and (2.96), for each $i = 1, \ldots, r$, we have $\bar{h}^i\big[\mathcal{S} \cap (\operatorname{Int}\sigma_i)\big] \subset \sigma_i$. Therefore, by (B.6), (2.97) implies

$$\mathcal{S}'_{q,r} \cap (\operatorname{Int}\tau) \subset \big[\mathcal{S} \cap (\operatorname{Int}\tau)\big] \cup \bigcup_{1\leq i\leq r,\, \tau\prec\sigma_i} \Big[\bar{h}^i\big(\mathcal{S} \cap (\operatorname{Int}\sigma_i)\big) \cap (\operatorname{Int}\tau)\Big]. \tag{2.103}$$

Claim: If $\ell = 1, \ldots, q - a$ then, letting $m = q - \ell$, for every $\tau \in Q^{(q-\ell)}$ we have

$$\mathcal{H}^a\big[\tilde{\mathcal{S}}_{q-\ell}\cap(\operatorname{Int}\tau)\big] \leq \sum_{j=1}^{\ell} \sum_{\tau\prec\tau_1\prec\cdots\prec\tau_j\in Q,\, \tau_1\in(Q^{(q-\ell)})^c} \phi^j\, \mathcal{H}^a\big[\mathcal{S}\cap(\operatorname{Int}\tau_j)\big]+\mathcal{H}^a\big[\mathcal{S}\cap(\operatorname{Int}\tau)\big]. \tag{2.104}$$

First, consider the case $\ell = 1$. Let $\tau \in Q^{(q-1)}$. Then by (2.103) and (2.48), we have

$$\begin{aligned}
\mathcal{H}^a\big[\tilde{\mathcal{S}}_{q-1} \cap (\operatorname{Int}\tau)\big] &= \mathcal{H}^a\big[\mathcal{S}'_{q,M_q} \cap (\operatorname{Int}\tau)\big] \\
&\leq \mathcal{H}^a\left[\big(\mathcal{S} \cap (\operatorname{Int}\tau)\big) \cup \left(\bigcup_{1\leq i\leq M_q,\, \tau\prec\sigma_i} \Big[\bar{h}^i\big(\mathcal{S} \cap (\operatorname{Int}\sigma_i)\big) \cap (\operatorname{Int}\tau)\Big]\right)\right] \\
&\leq \mathcal{H}^a\big[\mathcal{S} \cap (\operatorname{Int}\tau)\big] + \sum_{1\leq i\leq M_q,\, \tau\prec\sigma_i} \mathcal{H}^a\Big[\bar{h}^i\big(\mathcal{S} \cap (\operatorname{Int}\sigma_i)\big) \cap (\operatorname{Int}\tau)\Big] \\
&\leq \mathcal{H}^a\big[\mathcal{S} \cap (\operatorname{Int}\tau)\big] + \phi \sum_{1\leq i\leq M_q,\, \tau\prec\sigma_i} \mathcal{H}^a\big[\mathcal{S} \cap (\operatorname{Int}\sigma_i)\big].
\end{aligned} \tag{2.105}$$

This proves (2.104) in the case $\ell = 1$.

Inductively, let $k = 1, \ldots, q-a$ and suppose (2.104) holds for $\ell = 1, \ldots, k$. We show that it holds with $\ell = k+1$. First apply (2.104) with the $(q-k)$-skeleton, $Q^{(q-k)}$, in place of Q, $\tilde{\mathcal{S}}_{q-k}$ in place of $\mathcal{S}$, and $\ell = 1$. Let $\tau \in Q^{(q-k-1)}$. Then by (2.105),

$$\mathcal{H}^a\big[\tilde{\mathcal{S}}_{q-k-1} \cap (\operatorname{Int}\tau)\big] \leq \sum_{\tau\prec\tau_1\in\partial Q^{(q-k)}} \phi\, \mathcal{H}^a(\tilde{\mathcal{S}}_{q-k} \cap (\operatorname{Int}\tau_1)) + \mathcal{H}^a\big[\tilde{\mathcal{S}}_{q-k} \cap (\operatorname{Int}\tau)\big]$$

Apply the induction hypothesis with $\ell = k$, but re-index the τ_j's as follows: $\tau \to \tau_1$, $\tau_1 \to \tau_2, \ldots$, and $\tau_j \to \tau_{j+1}$. Then let $i = j+1$.

$$\begin{aligned}
\mathcal{H}^a\big[\tilde{\mathcal{S}}_{q-k-1} \cap (\operatorname{Int}\tau)\big] \leq \sum_{\tau\prec\tau_1\in\partial Q^{(q-k)}} \phi \Bigg[\sum_{i=2}^{k+1} \sum_{\tau_1\prec\cdots\prec\tau_i\in Q,\, \tau_2\in(Q^{(q-k)})^c} &\phi^{i-1}\mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\tau_i)) \\
&+ \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\tau_1))\Bigg] \\
&+ \mathcal{H}^a\big[\tilde{\mathcal{S}}_{q-k} \cap (\operatorname{Int}\tau)\big].
\end{aligned}$$

Apply the induction hypothesis to the last term with $\ell = k$ and note that $\tau_1 \in \partial Q^{(q-k)}$ and $\tau_1 \prec \tau_2$ automatically implies $\tau_2 \in (Q^{(q-k)})^c$.

$$\begin{aligned}
\mathcal{H}^a\big[\tilde{\mathcal{S}}_{q-k-1} \cap (\operatorname{Int}\tau)\big] \leq \sum_{\tau\prec\tau_1\in\partial Q^{(q-k)}} \Bigg[\sum_{i=2}^{k+1} \sum_{\tau_1\prec\cdots\prec\tau_i\in Q} &\phi^{i}\mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\tau_i)) \\
&+ \phi\mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\tau_1)\Bigg] \\
+ \sum_{j=1}^{k} \sum_{\tau\prec\tau_1\prec\cdots\prec\tau_j\in Q;\, \tau_1\in(Q^{(q-k)})^c} &\phi^{j}\mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\tau_j)) \\
&+ \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\tau)).
\end{aligned} \tag{2.106}$$

Suppose $\tau \prec \tau_1 \prec \cdots \prec \tau_j \in Q$, $j = k+1$, and $\tau_1 \in (Q^{(q-k)})^c$. Then $q \geq \dim\tau_j = \dim\tau_{k+1} \geq \dim\tau_1 + k$. Hence, $q - k \geq \dim\tau_1 \geq q-k+1$, an impossibility. Therefore, if

$j = k+1$, the following sum is empty:

$$\sum_{\tau \prec \tau_1 \prec \cdots \prec \tau_j \in Q;\, \tau_1 \in (Q^{(q-k)})^c} \phi^j \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int} \tau_j)) = 0.$$

Hence, by (2.106),

$$\begin{aligned} \mathcal{H}^a\big[\tilde{\mathcal{S}}_{q-k-1} \cap (\operatorname{Int} \tau)\big] \leq & \sum_{i=1}^{k+1} \sum_{\tau \prec \tau_1 \prec \cdots \prec \tau_i \in Q; \tau_1 \in \partial Q^{(q-k)}} \phi^i \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int} \tau_i)) \\ & + \sum_{j=1}^{k+1} \sum_{\tau \prec \tau_1 \prec \cdots \prec \tau_j \in Q;\, \tau_1 \in (Q^{(q-k)})^c} \phi^j \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int} \tau_j)) + \mathcal{H}^a\big[\mathcal{S} \cap (\operatorname{Int} \tau)\big]. \end{aligned}$$

Since $\partial Q^{(q-k)} \cap (Q^{(q-k)})^c = \varnothing$ and $\partial Q^{(q-k)} \cup (Q^{(q-k)})^c = (Q^{(q-k-1)})^c$ it follows that (2.104) holds with $\ell = k+1$. By induction (2.104) holds in general and the claim (2.104) is proven.

Let $\tau_0 \in \partial Q^{(a)}$. Then $\tau_0 \prec \tau_1 \in Q$ automatically implies $\tau_1 \in (Q^{(a)})^c$. Thus, applying (2.104) with $\ell = q - a$ and $\tau = \tau_0$, we can start the summation in (2.104) at $j = 0$ and get for every a-simplex $\tau_0 \in \partial Q^{(a)}$

$$\mathcal{H}^a\big[\tilde{\mathcal{S}}_a \cap (\operatorname{Int} \tau_0)\big] \leq \sum_{j=0}^{q-a} \sum_{\tau_0 \prec \tau_1 \prec \cdots \prec \tau_j \in Q} \phi^j \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int} \tau_j)), \quad \tau_0 \in \partial Q^{(a)}. \tag{2.107}$$

Next, let $\sigma \in (Q^{(a)})^c$ and let $\tau_0 \preccurlyeq \sigma$ be an a-simplex. We count the number of chains $\tau_0 \prec \tau_1 \prec \cdots \prec \tau_{j-1} \prec \tau_j = \sigma$. If $i = a$ let $j = 0$. For $i = a+1, a+2, \ldots$ let j be any element of $\{1, \ldots, i-a\}$. Let $V_0 = \{0, \ldots, a\}$. So V_0 has $a+1$ elements. Let N_j^i denote the number of filtrations $V_0 \subsetneqq \cdots \subsetneqq V_j := \{0, \ldots, i\}$. Note that for $k = 1, \ldots, j-1$, the elements of V_k do not have to be consecutive integers. Thus, V_k will often contain fewer than $\max V_k$ elements and if $0 < k_1 < k_2 < j$ we may have $\max V_{k_1} > k_2$. We have $N_0^a = 1$ and for $i \geq a+1$ we also have $N_1^i = 1$. Moreover, $N_j^i = 0$ if $j > i - a$. If $a+1 = t_0 < t_1 < \cdots < t_{j-1} < t_j = i+1$, the number of such filtrations s.t. the cardinality of $V_k = t_k$ $(k = 0, \ldots, j)$ is clearly the multinomial coefficient $\binom{i-a}{t_1 - t_0,\, t_2 - t_1,\, \cdots,\, t_j - t_{j-1}}$. Thus, if $i > a$ (so $j \geq 1$),

$$N_j^i = \sum_{a+1 < t_1 < \cdots < t_{j-1} < i+1} \binom{i-a}{t_1 - a - 1,\ t_2 - t_1,\ \cdots,\ i - t_{j-1} + 1} < j^{i-a}. \tag{2.108}$$

Let $\tau_0, \sigma \in Q$ with $\tau_0 \preccurlyeq \sigma$ and $\dim \tau_0 = a$. Then $N_j^{\dim \sigma}$ is the number of chains $\tau_0 \prec \tau_1 \prec \cdots \prec \tau_{j-1} \prec \tau_j = \sigma$.

Let $m = 0, 1, \ldots$. Let $\psi_m := 0$ if $m < a$, let $\psi_a := 1$, and for $m > a$ let

$$\psi_m := \psi_m(\phi) := \sum_{j=0}^{m-a} N_j^m \, \phi^j \text{ and } \psi = \psi(\phi, q) := \max\{\psi_1, \ldots, \psi_q\} \geq 1. \tag{2.109}$$

Note that by (2.90) and (2.108), ψ only depends on a, $q = \dim Q$, and $t_{min}(Q)$ (see (2.88)).

Now $\tilde{\mathcal{S}}_a \cap |Q| \subset |Q^{(a)}|$ and $\mathcal{H}^a(|Q_{a-1}|) = 0$. But

$$(\tilde{\mathcal{S}}_a \cap |Q|) \setminus \left(\bigcup_{\tau_0 \in \partial Q^{(a)}} \operatorname{Int} \tau_0 \right) \subset |Q_{a-1}|.$$

Moreover, by (B.5'), the sets $\operatorname{Int}\tau_0$ ($\tau_0 \in \partial Q^{(a)}$) are disjoint. Therefore,

$$\mathcal{H}^a(\tilde{\mathcal{S}}_a \cap |Q|) = \sum_{\tau_0 \in \partial Q^{(a)}} \mathcal{H}^a\big[\tilde{\mathcal{S}}_a \cap (\operatorname{Int}\tau_0)\big].$$

Hence, from (2.107) we get

$$\begin{aligned}\mathcal{H}^a(\tilde{\mathcal{S}}_a \cap |Q|) &\leq \sum_{\tau_0 \in \partial Q^{(a)}} \sum_{j=0}^{q-a} \sum_{\tau_0 \prec \tau_1 \prec \cdots \prec \tau_j \in Q} \phi^j \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\tau_j)) \\ &= \sum_{\tau_0 \in \partial Q^{(a)}} \sum_{j=0}^{q-a} \sum_{\tau_j \in Q} \phi^j \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\tau_j)) \sum_{\tau_0 \prec \tau_1 \prec \cdots \prec \tau_{j-1} \prec \tau_j} 1.\end{aligned}$$

Now let $\sigma = \tau_j \in Q$ and recall that $N_j^{\dim\sigma} = 0$ if $j > \dim\sigma - a$. Hence

$$\begin{aligned}\mathcal{H}^a(\tilde{\mathcal{S}}_a \cap |Q|) &\leq \sum_{\tau_0 \in \partial Q^{(a)}} \sum_{\tau_0 \preccurlyeq \sigma \in Q} \sum_{j=0}^{\dim\sigma - a} \phi^j N_j^{\dim\sigma}\, \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\sigma)) \\ &= \sum_{\tau_0 \in \partial Q^{(a)}} \sum_{\tau_0 \preccurlyeq \sigma \in Q} \psi_{\dim\sigma}\, \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\sigma)) \\ &\leq \sum_{\tau \in \partial Q^{(a)}} \sum_{\tau \preccurlyeq \sigma \in Q} \psi \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\sigma)) \\ &= \psi \sum_{\sigma \in Q;\, \dim\sigma \geq a} \sum_{\tau \in \partial Q^{(a)};\, \tau \preccurlyeq \sigma} \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\sigma)).\end{aligned}$$

Now, if $\dim\sigma \geq a$, the number of a-faces of σ is $\binom{\dim\sigma+1}{a+1}$ and if $\dim\sigma < a$ then $\mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\sigma)) = 0$. Therefore, by (B.7),

$$\begin{aligned}\mathcal{H}^a(\tilde{\mathcal{S}}_a \cap |Q|) &\leq \psi \sum_{\sigma \in Q;\, \dim\sigma \geq a} \binom{\dim\sigma + 1}{a+1} \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\sigma)) \\ &\leq \binom{q+1}{a+1} \psi \sum_{\sigma \in Q;\, \dim\sigma \geq a} \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\sigma)) \qquad (2.110) \\ &= \binom{q+1}{a+1} \psi \sum_{\sigma \in Q} \mathcal{H}^a(\mathcal{S} \cap (\operatorname{Int}\sigma)) \\ &= \binom{q+1}{a+1} \psi\, \mathcal{H}^a(\mathcal{S} \cap |Q|).\end{aligned}$$

Now, $\tilde{\mathcal{S}}_a \subset |Q^{(a)}|$, but $\tilde{\mathcal{S}}_a$ may not be a subcomplex of $Q^{(a)}$. I.e., there may exist partial simplices of $\mathcal{S}$ in $Q^{(a)}$. If this happens continue recursively pushing $\tilde{\mathcal{S}}_a$ out of such simplices. Recall $\mathcal{S}'_{a,0} = \tilde{\mathcal{S}}_a$ and M_a = the number of a-simplices in Q. Let $i = 0, \ldots, M_a - 1$, suppose one has constructed $\mathcal{S}'_{a,i}$, and let $\omega \in \partial Q^{(a)}$ be an a-simplex that has not yet benefited from the pushing operation, but suppose it is next in line. If ω is not a partial simplex of $\mathcal{S}'_{a,i}$ then $\mathcal{S}'_{a,i+1} = \mathcal{S}'_{a,i}$ so $\mathcal{H}^a(\mathcal{S}'_{a,i+1}) = \mathcal{H}^a(\mathcal{S}'_{a,i})$. If ω is a partial simplex of $\mathcal{S}'_{a,i}$ then by (points a, f, h) of lemma 2.1 we have $\mathcal{H}^a\big[(\operatorname{Int}\rho) \cap \mathcal{S}'_{a,i+1}\big] \leq \mathcal{H}^a\big[(\operatorname{Int}\rho) \cap \mathcal{S}'_{a,i}\big]$ for any $\rho \in \big(Q^{(a-1)}\big)^c$. As

for $\mathcal{H}^a(\rho \cap \mathcal{S}'_{a,i+1})$ with $\rho \in Q^{(a-1)}$, of course $\mathcal{H}^a\big[(\operatorname{Int}\rho) \cap \mathcal{S}'_{a,i+1}\big] = \mathcal{H}^a\big[(\operatorname{Int}\rho) \cap \mathcal{S}'_{a,i}\big] = 0$. In summary, further pushing from $\tilde{\mathcal{S}}_a$ can only reduce the $\mathcal{H}^a$-measure. Hence,

$$\mathcal{H}^a(\tilde{\mathcal{S}} \cap |Q|) = \mathcal{H}^a(\tilde{\mathcal{S}}_0 \cap |Q|) \leq \binom{q+1}{a+1} \psi\, \mathcal{H}^a(\mathcal{S} \cap |Q|) = K_2\big[\phi(Q), q\big] \mathcal{H}^a(\mathcal{S} \cap |Q|), \tag{2.111}$$

where

$$K_2 = K_2\big[\phi(Q), q\big] = \binom{q+1}{a+1} \psi\big[\phi(Q), q\big]. \tag{2.112}$$

(See (2.89) and (2.109).) By (2.109)

$$K_2 \geq 1. \tag{2.113}$$

Then (1.1) holds with $K = K_2$ if $0 < a < q$. (See (2.44).) (1.1) holds for general a if we take $K = \max\{K_1, K_2\}$, where K_1 is defined in (2.43).

2.6. **Proof of part (10) of theorem 1.1.** First of all, a subdivision of P is also a finite simplicial complex so parts 1 through 8 of theorem 1.1 automatically hold for any subdivision of P. The question is whether we can choose arbitrarily fine subdivisions of P and a single constant $K < \infty$ in (1.1) that will work for all those subdivisions.

Let $p = \dim P$. Suppose $0 < a < q$. By Munkres [Mun66, Lemma 9.4, p. 92], there exists $t_0 \in \big(0, t_{min}(Q)\big]$ (see (2.88)) depending only on P with the following property. For every $\epsilon > 0$ there exists a subdivision, $P' = P'(\epsilon)$, of P s.t. the diameter of the largest simplex in $P'(\epsilon)$ is no greater than ϵ and the thickness of every simplex (of positive dimension) in $P'(\epsilon)$ is at least t_0. By (2.82), (2.85), (2.87), (2.89), (2.109), and (2.112), we have $K_2\big[\phi(a,p,t_0), p\big] \geq K_2\big[\phi(Q), q\big]$. Hence, by (2.111), if we replace $K = K_2\big[\phi(Q), q\big]$ by $K = K_2\big[\phi(a,p,t_0), p\big]$ then (1.1) will hold if P is replaced by $P'(\epsilon)$.

As for the case $a = q$, all partial simplices in Q can be ignored because the pushing out operation repaces them by $(\mathcal{H}^a = \mathcal{H}^q)$-null sets. Only q-simplices lying in $\mathcal{S}$ matter. By points (d,h) of lemma 2.1, pushing out and subdivision does not affect the total volume of these simplices.

Finally, consider the case $a = 0$, so $\dim(\mathcal{S} \cap |Q|) = 0$. The argument given at the beginning of subsection 2.3 shows that (1.1) holds for any $K \geq 1$ for any complex so, in particular, it holds for any $P'(\epsilon)$.

Hence, part (10) of theorem 1.1 holds with $K = \max\{K_1, K_2\}$ (see (2.43)), providing we use the updated version of K_2 defined above. That concludes the proof of theorem 1.1.

APPENDICES

A. Miscellaneous proofs

Proof of corollary 1.2. Suppose P is a simplicial complex s.t. $|P| \subset \mathbb{R}^N$ and (1.2) holds. Then, by lemma B.4(i,ii), $|P|$ is a locally compact subspace of $\mathbb{R}^N$ and P is locally finite. We prove that parts (1) through (9) of theorem 1.1 continue to hold and assertion (10') also holds. Since P is locally finite and Q is finite, the complex P_Q is finite. The idea is to apply the theorem, as stated, to P_Q. Let $(\tilde{\mathcal{S}}_Q, \tilde{\Phi}_Q)$ be the set-map pair whose existence is asserted by the theorem (applied to P_Q and $\big(\mathcal{S} \cap |P_Q|, \Phi|_{|P_Q| \setminus \mathcal{S}}\big)$). (Here, $\Phi|_{|P_Q| \setminus \mathcal{S}}$ is the restriction of Φ to $|P_Q| \setminus \mathcal{S}$.) Thus, $\tilde{\mathcal{S}}_Q$ is a closed subset of $|P_Q|$ and $\tilde{\Phi}_Q : |P_Q| \setminus \tilde{\mathcal{S}}_Q \to \mathsf{F}$ is continuous. Now define

$$\tilde{\mathcal{S}} := \big(\mathcal{S} \setminus |P_Q|\big) \cup \tilde{\mathcal{S}}_Q \text{ and } \tilde{\Phi}(x) = \begin{cases} \Phi(x), & \text{if } x \in |P| \setminus |P_Q|, \\ \tilde{\Phi}_Q(x), & \text{if } x \in |P_Q|. \end{cases} \tag{A.1}$$

Obviously, by theorem 1.1, parts (2), (3), and (5) through (9) of theorem 1.1 still hold with this definition because P_Q is finite. (Proof of part (2) uses (C.5).)

First we show

$$\text{If } \sigma \in P \setminus P_Q \text{ then } \tilde{\mathcal{S}} \cap \sigma = \mathcal{S} \cap \sigma \text{ and } \tilde{\Phi}|_{\sigma \setminus \mathcal{S}} = \Phi|_{\sigma \setminus \mathcal{S}}. \tag{A.2}$$

By (A.1), this is obvious if $\sigma \in P \setminus P_Q$ with $\sigma \cap |P_Q| = \varnothing$. So let $\sigma \in P \setminus P_Q$ satisfy $\sigma \cap |P_Q| \neq \varnothing$. By (A.1) again, $\tilde{\mathcal{S}} \cap (\sigma \setminus |P_Q|) = \mathcal{S} \cap (\sigma \setminus |P_Q|)$ and $\tilde{\Phi}|_{\sigma \setminus (\mathcal{S} \cup |P_Q|)} = \Phi|_{\sigma \setminus (\mathcal{S} \cup |P_Q|)}$. As for $\tilde{\mathcal{S}} \cap \sigma \cap |P_Q|$, note that by (B.5), $\sigma \cap |P_Q|$ is the union of simplices in P_Q that are faces of σ. Let $\tau \in P_Q$ be such a simplex. If $\tau \cap |Q| \neq \varnothing$, then $\sigma \cap |Q| \neq \varnothing$. But this contradicts the assumption that $\sigma \notin P_Q$. Therefore, $\tau \cap |Q| = \varnothing$. Hence, by part (4) of theorem 1.1, $\tilde{\mathcal{S}} \cap \sigma \cap |P_Q| = \mathcal{S} \cap \sigma \cap |P_Q|$ and $\tilde{\Phi}|_{(\sigma \cap |P_Q|) \setminus \mathcal{S}} = \Phi|_{(\sigma \cap |P_Q|) \setminus \mathcal{S}}$. This completes the proof of (A.2).

That part (4) of the theorem still holds for P is an obvious consequence of (A.2).

We still need to check that $\tilde{\mathcal{S}}$ is closed and $\tilde{\Phi}$ is continuous on $|P| \setminus \tilde{\mathcal{S}}$, prove that part (1) of the theorem still holds, and verify that assertion (10') holds. By definition of the topology on $|P|$ (see appendix B), to show that $\tilde{\mathcal{S}}$ is closed it suffices to show that $\tilde{\mathcal{S}} \cap \sigma$ is closed in σ for every $\sigma \in P$. But by theorem 1.1, (A.2), and (A.1) whether $\sigma \in P_Q$ or $\sigma \in P \setminus P_Q$ since $\mathcal{S}$ is closed, $\tilde{\mathcal{S}} \cap \sigma$ is closed in σ. Thus, $\tilde{\mathcal{S}}$ is closed.

In a similar way we show that $\tilde{\Phi}$ is continuous on $|P| \setminus \tilde{\mathcal{S}}$. Let $U \subset \mathsf{F}$ be open. We must show that $\tilde{\Phi}^{-1}(U)$ is open in $|P| \setminus \tilde{\mathcal{S}}$. But $\tilde{\mathcal{S}}$ is closed so it suffices to show that $\tilde{\Phi}^{-1}(U)$ is open in $|P|$. Therefore, it suffices to show that $G(\sigma) := \tilde{\Phi}^{-1}(U) \cap \sigma$ is open in σ for every $\sigma \in P$. But this follows from theorem 1.1, (A.2), and the continuity of Φ on $|P| \setminus \mathcal{S}$.

Next, we prove that part (1) of theorem 1.1 still holds for P. Assume F has a metric, d, and Φ is locally Lipschitz on $|P| \setminus \mathcal{S}$. Let $x \in |P| \setminus \tilde{\mathcal{S}}$ be arbitrary. We need to find a neighborhood of x on which $\tilde{\Phi}$ is Lipschitz. Let P^Q be the subcomplex of P consisting of all simplices in P that are faces of simplices in $P \setminus P_Q$. So $P \setminus P_Q \subset P^Q$. Then, by (A.2) we have $\tilde{\mathcal{S}} \cap |P^Q| = \mathcal{S} \cap |P^Q|$ and that $\tilde{\Phi}$ and Φ are both defined and agree on $|P^Q| \setminus \mathcal{S}$. Thus, if $x \in |P| \setminus \big(|P_Q| \cup \tilde{\mathcal{S}}\big)$, $\tilde{\Phi}$ is Lipschitz in a neighborhood of x. By part (1) of the theorem, the same thing is true if $x \in |P| \setminus \big(|P^Q| \cup \tilde{\mathcal{S}}\big)$.

So assume $x \in \big(|P_Q| \cap |P^Q|\big) \setminus \tilde{\mathcal{S}}$. By (A.1) and assumption on Φ, there exists $K' < \infty$ and an open set $U \subset |P| \setminus \tilde{\mathcal{S}}$ s.t. $x \in U$ and $\tilde{\Phi}$ is Lipschitz on $U \cap |P^Q|$ with Lipschitz constant

$K' < \infty$, say. By theorem 1.1 part (1), we may assume $\tilde{\Phi}$ is Lipschitz on $U \cap |P_Q|$ and that the same Lipschitz constant K' works for $\tilde{\Phi}|_{U \cap |P_Q|}$.

By (B.7), there exists a unique simplex $\sigma \in P$, s.t. $x \in \operatorname{Int} \sigma$. By (B.8), $\operatorname{St} \sigma$ is an open neighborhood of x. Therefore, for $t \in (0,1)$ the following set is also an open neighborhood of x.

$$V_{t,x} := t\big((\operatorname{St} \sigma) - x\big) + x.$$

Here, the vector operations are performed point-wise. Pick $t > 0$ sufficiently small that $V_{t,x} \subset U$.

We show that $\tilde{\Phi}$ is Lipschitz on $V_{t,x}$. If $\omega \in P$, let

$$\omega_{t,x} := t(\omega - x) + x, \quad t \in (0,1). \tag{A.3}$$

Then, $\omega_{t,x}$ is a simplex and

$$\big\{\omega_{t,x} \subset \mathbb{R}^N : \omega \text{ is a face of some } \rho \in P \text{ s.t. } \sigma \subset \rho\big\}$$

is a finite simplicial complex, call it $P_{t,x}$. ($P_{t,x}$ is finite since P is locally finite.) If $\omega \in P$, then $\operatorname{Int} \omega \subset \operatorname{St} \sigma$ implies $\omega_{t,x} \in P_{t,x}$.

Let $y, z \in V_{t,x}$. Then there exist $\rho, \zeta \in P$ s.t. $\sigma \subset \rho \cap \zeta$ and $y \in \operatorname{Int} \rho_{t,x}$ and $z \in \operatorname{Int} \zeta_{x,t}$. The simplex ρ belongs to P_Q or P^Q and the same for ζ. Now by (A.2) and assumption on Φ, we have that $\tilde{\Phi}$ is Lipschitz on $V_{t,x} \cap |P^Q|$ with Lipschitz constant $K' < \infty$. And by theorem 1.1 part (1), $\tilde{\Phi}$ is Lipschitz on $V_{t,x} \cap |P^Q|$ also with Lipschitz constant K'. Hence, if both $\rho, \zeta \in P_Q$ or both $\rho, \zeta \in P^Q$ then

$$d\big[\tilde{\Phi}(y), \tilde{\Phi}(z)\big] \leq K'|y - z|. \tag{A.4}$$

Now $P = P_Q \cup P^Q$ so, in particular, (A.4) holds if $\rho \subset \zeta$ or $\zeta \subset \rho$. Without loss of generality (WLOG) we may assume $\rho \in P_Q$, $\zeta \in P^Q$, $\rho \not\subseteq \zeta$, and $\zeta \not\subseteq \rho$. We have $\rho_{t,x} \cap \zeta_{t,x} \neq \varnothing$ because $\sigma_{t,x} \subset \rho_{t,x} \cap \zeta_{x,t}$. Therefore, by (B.4) and (B.5), $\rho_{t,x} \cap \zeta_{x,t}$ is a simplex in $P_{x,t}$. Moreover, $\rho_{t,x} \cap \zeta_{t,x} \subset P_Q \cap P^Q$. Hence, by (A.4) and corollary B.3, there exists $K < \infty$ depending only on $\rho_{x,t}$ and $\zeta_{x,t}$ and $\tilde{x}, \tilde{y} \in \operatorname{Int}(\rho_{t,x} \cap \zeta_{x,t}) \subset |P_Q| \cap |P^Q|$ s.t.

$$\begin{aligned} d\big[\tilde{\Phi}(y), \tilde{\Phi}(z)\big] &\leq d\big[\tilde{\Phi}(y), \tilde{\Phi}(\tilde{y})\big] + d\big[\tilde{\Phi}(\tilde{y}), \tilde{\Phi}(\tilde{z})\big] + d\big[\tilde{\Phi}(\tilde{z}), \tilde{\Phi}(z)\big] \\ &\leq K'\big(|y - \tilde{y}| + |\tilde{y} - \tilde{z}| + |\tilde{z} - z|\big) \\ &\leq KK'|y - z|. \end{aligned}$$

Since the complex $P_{t,x}$ is finite, we can assume K works for any pair $\rho_{t,x}, \zeta_{t,x} \in P_{t,x}$ with $\operatorname{Int} \rho_{t,x}, \operatorname{Int} \zeta_{t,x} \subset V_{t,x}$. Thus, $\tilde{\Phi}$ is Lipschitz in $V_{t,x}$ and so part (1) of theorem 1.1, still holds for P.

Finally, we prove that (10') holds. The only issue is whether the subdivision of P_Q whose existence is asserted by part (10) of the theorem extends to a subdivision of all of P. But one can always use a "standard extension" (Munkres [Mun66, Definition 7.12, pp. 76–77]) for this purpose. (Munkres' [Mun66, Definition 7.1, p. 69] definition of simplicial complex includes the requirement (1.2).) □

Proof of corollary 1.5. Let $\epsilon > 0$. Define

$$A = \{x \in |P| : dist(x, \mathcal{S}) \geq \epsilon\}.$$

Using subdivisions permitted by theorem 1.1, part (10), we may assume all simplices of P have diameter $< \epsilon/2$. Let Q be the subcomplex of P consisting of all simplices that intersect $\mathcal{S}$, and all faces of all such simplices. Let $\sigma \in P$ have nonempty intersection with A. Suppose there exists $\tau \in Q$ s.t. $\sigma \cap \tau \neq \varnothing$. By definition of Q we may assume $\tau \cap \mathcal{S} \neq \varnothing$. Let $x_1 \in \mathcal{S} \cap \tau$, $x_2 \in \sigma \cap \tau$, and $x_3 \in \sigma \cap A$ Then

$$\epsilon \leq |x_1 - x_3| \leq |x_1 - x_2| + |x_2 - x_3| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon,$$

A contradiction. I.e.,

No simplex intersecting A is a face of any simplex intersecting $|Q|$.

Apply theorem 1.1 to Q. Point 1 of the corollary follows from parts (10 and 3) of the theorem. By part (7) of theorem 1.1, point (2) of the corollary holds. By part (4) of theorem 1.1, $\tilde{\Phi} = \Phi$ on A. I.e., point 3 of the corollary holds. Point 4 of the corollary follows from parts (10 and 8) of the theorem. Finally, point 5 of the corollary follows from part (10) of theorem 1.1. $\square$

Proof of lemma 2.1. The following technical lemma will come in handy.

Lemma A.1. *Recall the definition,* (2.32), *of* k. *We have*

$$k(\beta, \delta, t) \leq (1-\beta)^{\delta+t}, \text{ for } \beta \in [0,1], \ \delta + t > 0. \tag{A.5}$$

The gradient of k *is given by*

$$\nabla k(\beta, \delta, t) = -\left(\frac{\delta + t}{(1-\beta)^2}, \frac{\beta}{1-\beta}, \frac{\beta}{1-\beta} \right) \exp\left\{ -\frac{\beta(\delta+t)}{1-\beta} \right\}, \ \text{ for } \beta < 1. \tag{A.6}$$

Both k *and its first partial derivatives are continuous on* $(\beta, \delta, t) \in (-\infty, 1) \times \mathbb{R} \times \mathbb{R}$. *Let* $\ell = k$ *or* $\ell = \partial k/\partial \beta$ *or* $\ell = \partial k / \partial \delta$ *or* $\ell = \partial k/\partial t$. *If* $\delta > 0$ *and* $t' \geq 0$ *then*

$$\lim_{\beta \uparrow 1, t \downarrow 0} \ell(\beta, \delta, t) = \lim_{\beta \uparrow 1} \ell(\beta, \delta, t') = 0.$$

Similarly, if $\delta' \geq 0$ *and* $t > 0$ *then*

$$\lim_{\beta \uparrow 1, \delta \downarrow 0} \ell(\beta, \delta, t) = \lim_{\beta \uparrow 1} \ell(\beta', \delta, t) = 0.$$

It follows that k *is locally Lipschitz on*

$$T := \big[(-\infty, 1] \times [0, \infty) \times [0, \infty) \big] \ \setminus \ \big\{ (1, 0, 0) \big\}. \tag{A.7}$$

Note that $k(\beta, \delta, t) \in [0,1]$ *if* $(\beta, \delta, t) \in T$ *and* $\beta \in [0,1]$. *Note further that* T *is convex.*

Note that only one of conditions $\beta < 1$, $\delta > 0$, $t > 0$ need hold in order that $(\beta, \delta, t) \in (-\infty, 1] \times [0, \infty) \times [0, \infty)$ is actually in T.

Proof of lemma A.1. By definition of k, (A.5) holds with $\beta = 1$. So assume $0 \leq \beta < 1$. Since

$$k(\beta, \delta, t) = \exp\left\{ -\frac{\beta}{1-\beta} \right\}^{\delta + t},$$

WLOG we may take $\delta + t = 1$. Let

$$f(\beta) = \log(1-\beta) - \log k(\beta, \delta, t) = \log(1-\beta) + \frac{\beta}{1-\beta}.$$

Then

$$f(0) = 0 \text{ and } f'(\beta) = -\frac{1}{1-\beta} + \frac{1}{(1-\beta)^2}.$$

But the f' is nonnegative if $0 \leq \beta < 1$. (A.5) follows.

The existence and values of the limits follows from the fact that for $r \geq 0$

$$x^r e^{-x} \to 0 \text{ as } x \to +\infty.$$

By corollary C.4, k is locally Lipschitz on $(\beta, \delta, t) \in (-\infty, 1) \times \mathbb{R} \times \mathbb{R}$, but we also want to allow $\beta = 1$ so another approach is required. Let $x_1 = (\beta_1, \delta_1, t_1) \in T$ and $x_2 = (\beta_2, \delta_2, t_2) \in T$. Let $r > |x_2 - x_1|$. Since $x_1, x_2 \in T$, there exists $\epsilon > 0$ s.t.

$$\min\bigl\{(1-\beta_1) + \delta_1 + t_1,\ (1-\beta_2) + \delta_2 + t_2\bigr\} > \epsilon.$$

Let

$$K = \bigl\{x = (\beta, \delta, t) \in T : |x - x_1| \leq r \text{ and } (1-\beta) + \delta + t \geq \epsilon\bigr\}.$$

It is clear that T is convex. Therefore, K is convex, being the intersection of three convex sets. K is also compact and contains x_1 and x_2. k is continuous on K and its derivatives are continuous in the interior, K°, of K. Moreover, the derivatives of k can be extended continuously to all of K with finite values on the boundary of K. Hence, since K is compact, the derivatives of k are bounded in the interior of K. Let $D < \infty$ be an upper bound on the length of the gradient of k in K°. Suppose $\beta_1 = \beta_2 = 1$. Then, $\bigl|k(x_2) - k(x_1)\bigr| = 0 \leq D|x_2 - x_1|$. Suppose for definiteness that $\beta_1 < 1$. Consider the line segment, L, joining x_1 and x_2. It lies in K. Let f denote the restriction, $k|_D$, of k to L parametrized by arc length. Then f is differentiable in the interior of L (this is true even if x_1 and x_2 lie on the boundary of K, since $\beta_1 < 1$) and derivative of f is less than D in absolute value. Applying the Mean Value Theorem (Apostol [Apo57, Theorem 5–10, p. 93]) we see that $\bigl|k(x_2) - k(x_1)\bigr| \leq D|x_2 - x_1|$. $\square$

(Proof of lemma 2.1 continued.) We have

$$s(y,t) = \bar{h}(y) + k\bigl(b(y), \bar{\Delta}(y), t\bigr)\bigl[z - \bar{h}(y)\bigr], \quad y \in \operatorname{Int}\sigma,\ t \in [0,1]. \tag{A.8}$$

Let $y \in \sigma$ and $t \in [0,1]$. By (2.32) $0 \leq k\bigl(b(y), \bar{\Delta}(y), t\bigr) \leq 1$ so by (2.34),

$$s(y,t) \in \sigma, \quad \text{if } y \in \sigma,\ t \in [0,1]. \tag{A.9}$$

The following describes some properties of the function s defined in (2.34), $\bar{h}$ defined in (2.8), and f defined in (2.33).

Lemma A.2. *Except where noted, $y \in \sigma$ and $t \in [0,1]$ are arbitrary.*

(1) $s(y,0) = z$ *for every* $y \in \mathcal{C} \setminus (Bd\,\sigma)$.
(2) $s(z,t) = z$. *If* $s(y,t) = z$ *and* $t \in (0,1]$ *then* $y = z$.
(3) $s(y,t) \in Bd\,\sigma$ *if and only if* $y \in Bd\,\sigma$ *(in which case* $s(y,t) = y$*).*
(4) *If* $s(y,t) \neq z$*, then* $y \neq z$ *and* $\bar{h}\bigl[s(y,t)\bigr] = \bar{h}(y)$.
(5) $s(y,t) \in \mathcal{C}$ *if and only if* $y \in \mathcal{C}$.
(6) *There exists* $K'' = K''(z) < \infty$ *depending only on* σ *and* z*, and continuous in* z*, s.t. if* $y_1, y_2 \in \sigma$

$$|y_i - z|\bigl|\bar{h}(y_2) - \bar{h}(y_1)\bigr| \leq K''|y_2 - y_1|, \quad i = 1,2. \tag{A.10}$$

(7) *$\bar{h}(y)$ is locally Lipschitz in $y \in \sigma \setminus \{z\}$, $\bar{h}$ is Lipschitz on $\mathcal{A}$, and*

$$b(y) \text{ is Lipschitz on } \sigma. \tag{A.11}$$

(8) *The function $f : (y,t) \mapsto k\big(b(y), \bar{\Delta}(y), t\big)$ is a locally Lipschitz map on*

$$B_z := \big(\sigma \times [0,1]\big) \setminus \Big(\big[\mathcal{C} \cap (Bd\,\sigma)\big] \times \{0\}\Big). \tag{A.12}$$

(9) *s is locally Lipschitz on B_z.*

See below for the proof. The following is also proved below.

Lemma A.3. *For any $t \in (0,1]$ the map $y \mapsto s(y,t)$ is a one-to-one map of $Int\,\sigma$ onto itself. Moreover,*

$$\text{the map } \hat{s} : (y,t) \mapsto \big(s(y,t), t\big) \text{ on } (Int\,\sigma) \times (0,1] \text{ has a locally Lipschitz inverse.} \tag{A.13}$$

Let $(\mathrm{Lk}\,\sigma)^{(0)}$ denote the set of vertices in $\mathrm{Lk}\,\sigma$. Thus, every vertex in $\overline{\mathrm{St}}\,\sigma$ is either in $\sigma^{(0)}$ or $(\mathrm{Lk}\,\sigma)^{(0)}$. Let $x \in \overline{\mathrm{St}}\,\sigma$. Unless $x \in \mathrm{Lk}\,\sigma$, some of the $\beta_v(x)$'s are positive at $v \in \sigma^{(0)}$. In that case, define

$$\sigma(x) = \left(\sum_{v \in \sigma^{(0)}} \beta_v(x) \right)^{-1} \sum_{v \in \sigma^{(0)}} \beta_v(x)\, v \quad \in \ \sigma, \qquad x \in (\overline{\mathrm{St}}\,\sigma) \setminus (\mathrm{Lk}\,\sigma). \tag{A.14}$$

($\sigma(\cdot)$ will denote the function. σ without the parentheses will mean the simplex.) Then

$$\sigma(x) = x \text{ if } x \in \sigma \tag{A.15}$$

and by lemma B.2, corollary C.4, and (C.8), we have

$$\sigma(\cdot) \text{ is locally Lipschitz on } (\overline{\mathrm{St}}\,\sigma) \setminus (\mathrm{Lk}\,\sigma). \tag{A.16}$$

We have

$$\text{If } \rho \in P \text{ has } \sigma \text{ as a face (so } \rho \subset \overline{\mathrm{St}}\,\sigma) \text{ and } x \in \mathrm{Int}\,\rho, \text{ then } \sigma(x) \in \mathrm{Int}\,\sigma \tag{A.17}$$

because $x \in \mathrm{Int}\,\rho$ if and only if $\beta_v(x) > 0$ for every vertex v of ρ. In particular, $\beta_v(x) > 0$ for every vertex v of σ.

Partially conversely, suppose $\rho \in P$, $\rho \subset \overline{\mathrm{St}}\,\sigma$, and $\rho \cap \sigma \neq \varnothing$ but σ is not a face of ρ. (In particular, $\rho \neq \sigma$. Note that the case in which ρ is a proper face of σ is included.) Suppose $x \in \rho$ but $x \notin \mathrm{Lk}\,\sigma$. Then $\sum_{v \in \sigma^{(0)}} \beta_v(x) > 0$ but since σ is not a face of ρ, we have $\beta_v(x) = 0$ for some vertex v of σ. Say, $\beta_{v(j)}(x) = 0$. Then by (A.14) $\beta_{v(j)}\big[\sigma(x)\big] = 0$. To sum up, we have

$$\textit{If } \rho \in P,\ \rho \subset \overline{\mathrm{St}}\,\sigma, \text{ and } \rho \cap \sigma \neq \varnothing \text{ but } \sigma \text{ is not a face of } \rho \\ \textit{then } x \in \rho \setminus (\mathrm{Lk}\,\sigma) \text{ implies } \sigma(x) \in \mathrm{Bd}\,\sigma. \tag{A.18}$$

Suppose $\overline{\mathrm{St}}\,\sigma \neq \sigma$. (I.e., σ is a proper subset of $\overline{\mathrm{St}}\,\sigma$.) Then there exists $\rho \in P$ with $\rho \subset \overline{\mathrm{St}}\,\sigma$ having σ as a *proper* face. Let $\omega \subset \rho$ be the face of ρ "opposite" σ (appendix B). If $y \in \omega$ define $\sigma(y)$ to be an arbitrary point of σ. Then if $y \in \rho$ we can write

$$y = \mu\sigma(y) + (1-\mu)w \tag{A.19}$$

where $\mu \in [0,1]$ and $w \in \omega$. Now, suppose $y = \nu x + (1-\nu)u$, where $\nu \in [0,1]$, $x \in \sigma$, and $u \in \omega$. Then, by the geometric independence (appendix B) of the vertices of ρ we have $\nu = \mu$. If $y \notin \operatorname{Lk}\sigma$ then we also have $x = \sigma(y)$. If $y \notin \sigma$, then we also have $u = w(y)$. (In particular, μ and w are uniquely determined by $y \in \operatorname{Int}\rho$.) So we can define $\mu(y) = \mu$ and, providing $y \notin \sigma$, we can define $w(y) = w$. Adopt the following conventions.

$$w(y) = \begin{cases} \text{an arbitrary fixed point of } \operatorname{Lk}\sigma, & \text{if } y \in \sigma \text{ and } \overline{\operatorname{St}}\,\sigma \neq \sigma, \\ \text{and arbitrary fixed point of } |P|, & \text{if } \overline{\operatorname{St}}\,\sigma = \sigma. \end{cases} \tag{A.20}$$
$$\text{If } y \in \operatorname{Lk}\sigma, \text{ let } \sigma(y) \text{ be an arbitrary fixed point of } \sigma.$$

Clearly,

$$w(y) = \frac{1}{\sum_{v \in (\operatorname{Lk}\sigma)^{(0)}} \beta_v(y)} \sum_{v \in (\operatorname{Lk}\sigma)^{(0)}} \beta_v(y) v, \text{ if } y \in (\overline{\operatorname{St}}\,\sigma) \setminus \sigma, \text{ and} \tag{A.21}$$
$$\mu(y) = \sum_{v \in \sigma^{(0)}} \beta_v(y), \text{ if } y \in \overline{\operatorname{St}}\,\sigma.$$

It follows that if there are multiple simplices $\rho \in P$ with $\rho \subset \overline{\operatorname{St}}\,\sigma$ having σ as a proper face, and s.t. $y \in \rho$ then $\mu(y)$ and $w(y)$ are the same no matter which of these ρ's we use to define them. We then have

$$y \in \operatorname{Lk}\sigma \text{ if and only if } w(y) = y \text{ if and only if } \mu(y) = 0. \tag{A.22}$$
$$y \in \sigma \text{ if and only if } \mu(y) = 1 \text{ if and only if } \sigma(y) = y.$$

Since by lemma B.2, $\{\beta_v(y),\ v \in P^{(0)}\}$ is Lipschitz in y and $\mu(y)$ and $w(y)$ can be expressed in terms of barycentric coordinates, it follows that

$$\mu \text{ is Lipschitz on } \overline{\operatorname{St}}\,\sigma \text{ and } w \text{ is locally Lipschitz on } (\overline{\operatorname{St}}\,\sigma) \setminus \sigma. \tag{A.23}$$

Recall (2.35) and (2.36). If $\overline{\operatorname{St}}\,\sigma = \sigma$ define $\mu(y) = 1$ for every $y \in \overline{\operatorname{St}}\,\sigma = \sigma$ and let $w(y)$ be an arbitrary fixed point of $|P|$. Then we see that (2.35) still makes sense if $\overline{\operatorname{St}}\,\sigma = \sigma$. By (A.22) we have

$$g(y) = s(y,0), \text{ if } y \in \sigma \setminus \big[\mathcal{C} \cap (\operatorname{Bd}\sigma)\big].$$

Thus, whether $\overline{\operatorname{St}}\,\sigma$ equals σ or not we have

$$g(y) = g_z(y) = \tag{A.24}$$
$$\begin{cases} \mu(y)\, s_z\big(\sigma(y), 1-\mu(y)\big) + \big(1-\mu(y)\big)w(y) \in (\overline{\operatorname{St}}\,\sigma) \setminus \sigma, & \text{if } y \in \overline{\operatorname{St}}\,\sigma \setminus \sigma, \\ s(y,0) = \mu(y)\, s_z\big(\sigma(y), 1-\mu(y)\big) + \big(1-\mu(y)\big)w(y) \in \sigma, & \text{if } y \in \sigma \setminus \big[\mathcal{C} \cap (\operatorname{Bd}\sigma)\big], \\ y, & \text{if } y \in |P| \setminus (\overline{\operatorname{St}}\,\sigma). \end{cases}$$

(See (A.20).)

Note that, by lemma A.2(part 3), (A.19), and (A.22)

$$g(y) = y \text{ if:}$$
$$\begin{aligned} & y \in (\overline{\operatorname{St}}\,\sigma) \setminus \big[\mathcal{C} \cap (\operatorname{Bd}\sigma)\big] \text{ but } \sigma(y) \in \operatorname{Bd}\sigma, \\ & y \in \operatorname{Lk}\sigma \text{ (i.e., } \mu(y) = 0), \text{ or} \\ & y \in |P| \setminus (\overline{\operatorname{St}}\,\sigma). \end{aligned} \tag{A.25}$$

Claim:

$$\text{If } \rho \in P \text{ then } g(\rho \setminus \sigma) \subset \rho \setminus \sigma. \tag{A.26}$$

To see this, let $\rho \in P$ and $y \in \rho \setminus \sigma$. If $y \notin \overline{\mathrm{St}}\,\sigma$ then by (A.25), $g(y) = y \in \rho \setminus \sigma$. So suppose $y \in (\overline{\mathrm{St}}\,\sigma) \setminus \sigma$. ρ has a face ξ s.t. $\xi \subset \overline{\mathrm{St}}\,\sigma$ and $y \in \xi$. If $y \in \mathrm{Lk}\,\sigma$, then, again by (A.25), $g(y) = y \in \rho \setminus \sigma$. So suppose $y \notin \mathrm{Lk}\,\sigma$. Then $\xi \cap \sigma \neq \varnothing$. If σ is not a face of ξ then, by (A.18) and (A.25), we have $g(y) = y \in \rho \setminus \sigma$. Finally, suppose σ is a face of ξ (and hence of ρ). Then, by (A.9) and (A.24), we have $g(y) \in \xi \subset \rho$. But $y \notin \sigma$. Therefore, by (A.22), $\mu(y) < 1$. So, by (A.24) again, $\mu\big[g(y)\big] < 1$. So, by (A.22) again, $g(y) \in \rho \setminus \sigma$. This proves the claim (A.26).

By (A.18) and (A.25), g is the identity on all simplices in P that do not have σ as a face (so in particular do not equal σ). Moreover, by (A.22) and lemma A.2(part 1), $g(\mathcal{C} \setminus \mathrm{Bd}\,\sigma) = \{z\}$.

Claim:

$$g \text{ is a continuous map of } |P| \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big] \text{ into itself.} \tag{A.27}$$

First, we show $g : |P| \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big] \to |P| \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big]$. Suppose $y \in |P| \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big]$. If $y \notin \overline{\mathrm{St}}\,\sigma$, then $g(y) = y \notin \mathcal{C} \cap (\mathrm{Bd}\,\sigma)$. If $y \in (\overline{\mathrm{St}}\,\sigma) \setminus \sigma$, then, by (A.22), $\mu(y) < 1$, so $g(y) \notin \sigma \supset \mathcal{C} \cap (\mathrm{Bd}\,\sigma)$. Suppose $y \in \sigma \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big]$. Then, by (A.22), $\mu(y) = 1$, $\sigma(y) = y$, and by lemma A.2(parts 3, 5), $s\big(\sigma(y), 1 - \mu(y)\big) = s(y, 0) \notin \mathcal{C} \cap (\mathrm{Bd}\,\sigma)$. I.e., $g : |P| \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big] \to |P| \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big]$.

By lemma A.2(part 9), (A.23), and (A.16), g is continuous on $(\overline{\mathrm{St}}\,\sigma) \setminus (\sigma \cup (\mathrm{Lk}\,\sigma))$ (or $(\overline{\mathrm{St}}\,\sigma) \setminus \{\sigma \cup (\mathrm{Lk}\,\sigma)\} = \varnothing$). Let $y \in \sigma \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big]$ so, by (A.15), $\sigma(y) = y$. Let $\{y_m\} \subset |P|$ be a sequence converging to y. It suffices to consider two separate cases.

Suppose $\{y_m\} \subset |P| \setminus (\overline{\mathrm{St}}\,\sigma)$. Since P is finite WLOG, there exists $\rho \in P$ s.t. $\{y_m\} \subset \rho$. Hence, if $y \in \mathrm{Int}\,\sigma$ then, by (B.6), σ is a face of ρ. This contradicts $\{y_m\} \subset |P| \setminus (\overline{\mathrm{St}}\,\sigma)$. Therefore, $y \in (\mathrm{Bd}\,\sigma) \setminus \mathcal{C}$. Therefore, $\sigma(y) = y \in \mathrm{Bd}\,\sigma$ and, by (A.25), $g(y_m) = y_m \to y = g(y)$.

Next, suppose $\{y_m\} \subset \overline{\mathrm{St}}\,\sigma$. Then eventually $y_m \in (\mathrm{St}\,\sigma) \setminus (\mathrm{Lk}\,\sigma)$. By (A.16), (A.23), and (A.22), $y_m \to y \in \sigma \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big]$ implies $\mu(y_m) \to 1 = \mu(y)$ and $\sigma(y_m) \to \sigma(y) = y$. Hence, eventually, $\sigma(y_m) \notin \mathcal{C} \cap (\mathrm{Bd}\,\sigma)$. Thus, $g(y_m) \to y$ by lemma A.2 (part 9).

It suffices, finally, to consider $y \in \mathrm{Lk}\,\sigma$. Let $\{y_m\} \subset |P|$ be a sequence converging to y. By (A.24) and (A.25) it suffices to prove that if $\{y_m\} \subset (\overline{\mathrm{St}}\,\sigma) \setminus (\mathrm{Lk}\,\sigma)$ then $g(y_m) \to y$. By and (A.22), $\mu(y_m) \to 0$ and $w(y_m) \to y$ as $m \to \infty$. Since the range of s is the bounded set σ, we have $g(y_m) \to y = g(y)$ as $m \to \infty$. This completes the proof of the claim (A.27).

We have the following. (The proof is given below.)

Lemma A.4. *g is a locally Lipschitz map on $|P| \setminus \big[\mathcal{C} \cap (Bd\,\sigma)\big]$.*

Note further that, by (A.22) and (A.24),

$$g\Big(\sigma \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big]\Big) \subset \sigma \text{ and } g\big(|P| \setminus \sigma\big) \subset |P| \setminus \sigma. \tag{A.28}$$

It follows from this and the definition of $\mathcal{S}'$, (2.38), that

$$\mathcal{S}' \cap \sigma = \mathcal{C} \cap (\mathrm{Bd}\,\sigma). \tag{A.29}$$

This is the first sentence of point (a) of the lemma. The second sentence is then immediate from (2.6). By (2.11) and lemma A.2 (part 7) $\mathcal{C} \cap (\mathrm{Bd}\,\sigma)$ is a Lipschitz image of $\mathcal{A} = \mathcal{S} \cap \sigma$. Point (a) then follows from lemma C.2.

Let $\rho \in P$ not be a face of σ. Note that if $y \in \mathrm{Int}\,\rho$, then by (B.6), $y \notin \mathcal{C} \cap (\mathrm{Bd}\,\sigma)$ so $g(y)$ is defined. *First, suppose ρ has σ as a proper face* and let ω be the face of ρ opposite to σ. (In

particular, $\rho \subset \overline{\mathrm{St}}\,\sigma$.) Let $y \in \mathrm{Int}\,\rho$, then by (B.2) and (A.21), we have $w(y) \in \mathrm{Int}\,\omega$ and, by (A.17), we have $\sigma(y) \in \mathrm{Int}\,\sigma$. Hence, by lemma A.2(part 3) we have $s\big(\sigma(y), 1-\mu(y)\big) \in \mathrm{Int}\,\sigma$. Moreover, by (A.22), $1 > 1-\mu(y) > 0$. Therefore, by (A.24), $g(y) \in \mathrm{Int}\,\rho$.

Conversely, let $y' \in \mathrm{Int}\,\rho$. We show that there exists $y \in \mathrm{Int}\,\rho$ s.t. $y' = g(y)$. Write $y' = \mu(y')\sigma(y') + \big(1-\mu(y')\big)w(y')$. Since $y' \in \mathrm{Int}\,\rho$, we have, as before, $w(y') \in \mathrm{Int}\,\omega$, $\sigma(y') \in \mathrm{Int}\,\sigma$, and $1 > 1-\mu(y') > 0$. By lemma A.3, there exists a unique $x \in \mathrm{Int}\,\sigma$ s.t. $s\big(x, 1-\mu(y')\big) = \sigma(y')$. Let $y = \mu(y')x + \big(1-\mu(y')\big)w(y')$. Then $y \in \mathrm{Int}\,\rho$ and $\mu(y) = \mu(y')$, $\sigma(y) = x$, and $w(y) = w(y')$. Thus, by (A.24), $y' = \mu(y')s\big(x, 1-\mu(y')\big) + \big(1-\mu(y')\big)w(y) = g(y)$, as desired. This proves that $g(\mathrm{Int}\,\rho) = \mathrm{Int}\,\rho$. Now, as observed following (A.19), we have that $\mu(y)$ ($\mu(y')$) and $w(y)$ (resp. $w(y')$) are uniquely determined by $y \in \mathrm{Int}\,\rho$ (respectively [resp.], $y' \in \mathrm{Int}\,\rho$). Therefore, by lemma A.3, $y \in \mathrm{Int}\,\rho$ s.t. $g(y) = y'$ is uniquely determined by $y' \in \mathrm{Int}\,\rho$.

Let y, y' be as in the last paragraph. Now, $\mu(y')$, $\sigma(y')$, and $w(y')$ are uniquely determined by $y' \in \mathrm{Int}\,\rho$ in a locally Lipschitz fashion (by (A.16) and (A.23)). Hence, by (A.13) and (C.8), $x = x(y') \in \mathrm{Int}\,\sigma$ solving $s\big(x, 1-\mu(y')\big) = \sigma(y')$ is locally Lipschitz in $y' \in \mathrm{Int}\,\rho$. Therefore, by (C.8) again, we have that $y = \mu(y')x + \big(1-\mu(y')\big)w(y') \in \mathrm{Int}\,\rho$ solving $g(y) = y'$ is uniquely determined by $y' \in \mathrm{Int}\,\rho$ in a locally Lipschitz fashion. To sum up we have the following.

$$\text{(A.30)} \quad g \text{ is one-to-one on } \mathrm{Int}\,\rho,\ g(\mathrm{Int}\,\rho) = \mathrm{Int}\,\rho, \\ \text{and the restriction, } g|_{\mathrm{Int}\,\rho}, \text{ has a locally Lipschitz inverse on } \mathrm{Int}\,\rho.$$

Next, assume $\rho \in P$ is not a face of σ but neither is σ a face of ρ. If $\rho \subset \mathrm{Lk}\,\sigma$ then by (A.25), we have $g(y) = y$ for $y \in \mathrm{Int}\,\rho$. Suppose $\rho \subset \overline{\mathrm{St}}\,\sigma$ but $\rho \not\subset \mathrm{Lk}\,\sigma$ and let $y \in \mathrm{Int}\,\rho$. Then, by (B.6), $y \notin \mathrm{Lk}\,\sigma$, so, by (A.18), $\sigma(y) \in \mathrm{Bd}\,\sigma$. Therefore, by (A.25) again, we have $g(y) = y$ for $y \in \mathrm{Int}\,\rho$. Suppose $\rho \not\subset \overline{\mathrm{St}}\,\sigma$ and let $y \in \mathrm{Int}\,\rho$. Then by (B.6), $y \notin \overline{\mathrm{St}}\,\sigma$ so, yet again by (A.25), we have $g(y) = y$. Hence, (A.30) holds for ρ s.t. $\rho \subset \mathrm{Lk}\,\sigma$ or $\rho \not\subset \overline{\mathrm{St}}\,\sigma$.

It remains to prove the following. *Claim:*

$$\text{(A.31)} \qquad \text{If } \rho \in P \text{ is not a face of } \sigma \text{ then } g^{-1}(\mathrm{Int}\,\rho) = \mathrm{Int}\,\rho.$$

To see this let $\rho \in P$ not be a face of σ. By (A.30), we have $\mathrm{Int}\,\rho \subset g^{-1}(\mathrm{Int}\,\rho)$. Let $y \in \mathrm{Int}\,\rho$. By (B.6) $y \notin \sigma$. Suppose $x \in |P| \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big]$ and $g(x) = y$. Then by (A.28), $x \notin \sigma$. Let $\tau \in P$ satisfy $x \in \mathrm{Int}\,\tau$. (See (B.7).) Since $x \notin \sigma$, τ is not a face of σ. Therefore, by (A.30), $y = g(x) \in \mathrm{Int}\,\tau$. Thus, $(\mathrm{Int}\,\tau) \cap (\mathrm{Int}\,\rho) \neq \varnothing$. Hence, by (B.5'), $\tau = \rho$. In particular, $x \in \mathrm{Int}\,\rho$. This proves claim (A.31). Point (b) of the lemma follows.

We prove point (c) of the lemma, *viz.*, that $\mathcal{S}'$ is closed and Φ' is continuous off $\mathcal{S}'$ (and locally Lipschitz, too, when appropriate). *First, we prove that $\mathcal{S}'$ is closed.* Since $\mathcal{S}$ is closed, $\mathcal{S}' \setminus \sigma$ is closed in $|P| \setminus \sigma$ by (2.38) and (A.28). Moreover, by (2.6), $\mathcal{C}_z \cap (\mathrm{Bd}\,\sigma)$ is closed in $|P|$. So to prove $\mathcal{S}'$ is closed it suffices to show that all limit points of $\mathcal{S}' \setminus \sigma$ in σ lie in $\mathcal{C}_z \cap (\mathrm{Bd}\,\sigma)$. It suffices to show that if $x \in \sigma \setminus (\mathcal{C}_z \cap (\mathrm{Bd}\,\sigma))$ then x has an open neighborhood in $|P|$ disjoint from $\mathcal{S}'$.

Suppose first that $x \in \sigma \setminus \mathcal{C}_z$. We show that x has an open neighborhood disjoint from $\mathcal{S}'$. By lemma A.2(part 5) and (A.24) we have $g(x) = s(x, 0) \in \sigma \setminus C_z$. But $\mathcal{S}$ is closed and, by (2.6), $\mathcal{S} \cap \sigma \subset \mathcal{C}_z$ and $\mathcal{C}_z$ is also closed. I.e., $g(x)$ has a neighborhood V in $|P|$ disjoint from $\mathcal{S} \cup \mathcal{C}_z$. By (A.27) and (2.38), $g^{-1}(V)$ is a relatively open subset of the open set $|P| \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big]$ and $g^{-1}(V)$ is disjoint from $\mathcal{S}'$. Thus, $g^{-1}(V)$ is an open neighborhood of x disjoint from $\mathcal{S}'$.

Next, suppose $x \in \mathcal{C} \setminus (\mathrm{Bd}\,\sigma)$. By (2.4) the point z has a neighborhood U disjoint from $\mathcal{S} \cup (\mathrm{Bd}\,\sigma)$. By lemma A.4, the set $g^{-1}(U)$ is a relatively open subset of the open set $\subset$

$|P| \setminus \big[\mathcal{C} \cap (\operatorname{Bd} \sigma)\big]$. Hence, $g^{-1}(U)$ is open and, by (2.38) and (A.28), disjoint from $\mathcal{S}' \setminus \sigma = g^{-1}\big(\mathcal{S} \setminus \sigma\big)$. Moreover, by point (a) of the lemma, $\mathcal{S}' \cap \sigma$ is disjoint from the domain of g. *A fortiori,* $g^{-1}(U) \cap \mathcal{S}' \cap \sigma = \varnothing$. Thus, $g^{-1}(U) \cap \mathcal{S}' = \varnothing$. But, by (A.24) and lemma A.2(1), $g(x) = s(x, 0) = z \in U$. I.e., $x \in g^{-1}(U) \subset |P| \setminus \mathcal{S}'$. This completes the proof that $\mathcal{S}'$ is closed.

Next, we continue the proof of point (c) by *showing that Φ' is defined and continuous off $\mathcal{S}'$*. Let $x_0 \in |P| \setminus \mathcal{S}'$. First, suppose $x_0 \in |P| \setminus \sigma$. By (A.28), we have $g(x_0) \in |P| \setminus \sigma$. Then, by (2.37), $\Phi'(x_0)$ is defined except, perhaps, if $g(x_0) \in \mathcal{S} \setminus \sigma$, i.e., by (2.38), except, perhaps, if $x_0 \in \mathcal{S}'$. But $x_0 \in |P| \setminus \mathcal{S}'$. Therefore,

$$\text{if } x_0 \in |P| \setminus (\mathcal{S}' \cup \sigma), \text{ then } g(x_0) \notin \mathcal{S} \setminus \sigma \text{ (so } \Phi'(x_0) \text{ is defined).} \tag{A.32}$$

By (A.32) and (A.28), if $x_0 \in |P| \setminus (\mathcal{S}' \cup \sigma)$ then $g(x_0) \notin \mathcal{S}$, so Φ is defined and continuous at $g(x_0)$, and, by (A.27), g is continuous at x_0. But, by (2.37), $\Phi'(x_0) = \Phi \circ g(x_0)$. Therefore, Φ' is defined and continuous at x_0.

Next, let $x_0 \in \sigma \setminus \mathcal{S}' = \sigma \setminus \big(\mathcal{C} \cap (\operatorname{Bd} \sigma)\big)$ (see (A.29)). By (A.27) g is continuous at x_0. Therefore, by (2.37) it suffices to show that $g(x_0) \notin \mathcal{S}$. Since $x_0 \in \sigma$ we have by (A.24) $g(x_0) = s(x_0, 0)$. Suppose $x_0 \in \sigma \setminus \mathcal{C}$. By lemma A.2(part 5) we have $g(x_0) = s(x_0, 0) \in \sigma \setminus \mathcal{C} \subset \sigma \setminus \mathcal{S}$, by (2.6). Next, suppose $x_0 \in (\operatorname{Int} \sigma) \cap \mathcal{C}$. By lemma A.2(part 1) and (2.4) we have $g(x_0) = s(x_0, 0) = z \notin \mathcal{S}$. To sum up,

$$g(\sigma \setminus \mathcal{S}') \subset \sigma \setminus \mathcal{S}. \tag{A.33}$$

Thus, Φ is also continuous on $\sigma \setminus \mathcal{S}'$.

(A.32) and (A.33) tell us that $g\big[|P| \setminus \mathcal{S}'\big] \subset |P| \setminus \mathcal{S}$. By lemma A.4 and (C.8), if F (recall $\Phi : |P| \setminus \mathcal{S} \to \mathsf{F}$) is a metric space and Φ is locally Lipschitz on $|P| \setminus \mathcal{S}$, then $\Phi' = \Phi \circ g$ is locally Lipschitz on $|P| \setminus \mathcal{S}' \subset |P| \setminus \big[\mathcal{C}_z \cap (\operatorname{Bd} \sigma)\big]$. This completes the proof of point (c).

Suppose $\rho \in P$ and $\operatorname{Int} \rho \subset \mathcal{S}$. Hence, by (2.1), we have $\rho \neq \sigma$. Since $\mathcal{S}$ is closed, we have, in fact, that $\rho \subset \mathcal{S}$. Thus, if $\rho \subset \sigma$, then $\rho \subset \operatorname{Bd} \sigma$. By (A.29), we have $\mathcal{S}' \cap (\operatorname{Bd} \sigma) = \mathcal{C} \cap (\operatorname{Bd} \sigma)$. Hence, by (2.6), we have $\rho \subset \mathcal{S} \cap (\operatorname{Bd} \sigma) \subset \mathcal{C} \cap (\operatorname{Bd} \sigma) \subset \mathcal{S}'$. Next, assume $\rho \not\subseteq \sigma$, so by (B.6), $(\operatorname{Int} \rho) \cap \sigma = \varnothing$. Then by point (b) and (2.38) we have $\operatorname{Int} \rho \subset \mathcal{S}'$. This proves point (d) of the lemma.

Let $\tau \in P$ and suppose $\tau \cap \sigma = \varnothing$. Then $\tau \subset (\operatorname{Lk} \sigma) \cup \big(|P| \setminus (\overline{\operatorname{St}}\, \sigma)\big)$. Hence, by (A.25), if $y \in \tau$ then $g(y) = y$. Moreover, certainly τ is not a face of σ. Nor is any face of τ a face of σ. Hence, by (A.31), we have $g^{-1}(\tau) = \tau$. Therefore, if $y \in \tau$, then $y \in \mathcal{S}'$ if and only if $y \in \mathcal{S}$ and $\Phi' = \Phi$ on $\tau \setminus \mathcal{S}$. Point (e) of the lemma follows.

Now we prove point l of the lemma. However, *that point does not seem to be used anywhere!* Similarly, suppose $\overline{\operatorname{St}}\, \sigma = \sigma$. Let $y \in |P| \setminus \sigma = |P| \setminus (\overline{\operatorname{St}}\, \sigma)$. Then by (2.36) and (A.28), $g\big(\{y\}\big) = \{y\} = g^{-1}(y)$. Hence, by (2.38), $\mathcal{S}' \setminus \sigma = \mathcal{S} \setminus \sigma$ and, by (2.37), $\Phi'(y) = \Phi(y)$. It then follows from (A.29) and (2.6) that

$$|P| \setminus \big[(\operatorname{Int} \sigma) \cup \mathcal{S}'\big] \subset |P| \setminus \big[(\operatorname{Int} \sigma) \cup \mathcal{S}\big].$$

Suppose $y \in (\operatorname{Bd} \sigma) \setminus \big[\mathcal{C}_z \cap (\operatorname{Bd} \sigma)\big] = (\operatorname{Bd} \sigma) \setminus \mathcal{S}'$. (See (A.29).) Then, by (A.15), we have $\sigma(y) = y \in \operatorname{Bd} \sigma$ so, by (A.25), again $g(y) = y$. Thus, $\Phi'(y) = \Phi(y)$ and point (l) of the lemma is proved.

As for point (f), let $\tau \neq \sigma$ be a simplex in P of dimension no greater than n. *Claim:*

$$\mathcal{S}' \cap (\tau \setminus \sigma) = g^{-1}(\mathcal{S} \setminus \sigma) \cap (\tau \setminus \sigma) = g^{-1}\big[\mathcal{S} \cap (\tau \setminus \sigma)\big]. \tag{A.34}$$

The first equality is immediate from (2.38). As for the second equality, first suppose $y \in g^{-1}(\mathcal{S} \setminus \sigma) \cap (\tau \setminus \sigma)$. Then, there exists $x \in \mathcal{S} \setminus \sigma$ s.t. $x = g(y)$. Since $y \in \tau \setminus \sigma$, we have by (A.26) that $x \in \tau \setminus \sigma$. I.e., $g^{-1}(\mathcal{S} \setminus \sigma) \cap (\tau \setminus \sigma) \subset g^{-1}\big[\mathcal{S} \cap (\tau \setminus \sigma)\big]$. Conversely, suppose $x \in \mathcal{S} \cap (\tau \setminus \sigma)$ and let $y \in |P| \setminus \big[\mathcal{C} \cap (\operatorname{Bd} \sigma)\big]$ satisfy $g(y) = x$. Since $x \notin \sigma$, by (A.28), we have $y \in |P| \setminus \sigma$. By (B.7) there exists $\xi \in P$ s.t. $y \in \operatorname{Int} \xi$. Then ξ is not a face of σ. Since $y \in \operatorname{Int} \xi$, by point (b), we have $x \in \operatorname{Int} \xi$. But $x \in \mathcal{S} \cap (\tau \setminus \sigma)$. Hence, by (B.6), ξ is a face of τ. I.e., $y \in \tau \setminus \sigma$. I.e., $g^{-1}\big[\mathcal{S} \cap (\tau \setminus \sigma)\big] \subset g^{-1}(\mathcal{S} \setminus \sigma) \cap (\tau \setminus \sigma)$ and the claim (A.34) is proved.

If $\tau \subset \operatorname{Lk} \sigma$, then $\mathcal{S}' \cap (\tau \setminus \sigma) = \mathcal{S} \cap (\tau \setminus \sigma)$ and $\Phi' = \Phi$ on $\tau \setminus (\sigma \cup \mathcal{S}')$, by point (e). If $\tau \subset \overline{\operatorname{St}}\, \sigma$ but $\tau \nsubseteq \operatorname{Lk} \sigma$, then $\tau \cap \sigma \neq \varnothing$. Since $\dim \tau \leq n = \dim \sigma$ and $\tau \neq \sigma$, the simplex σ cannot be a face of τ. If $y \in \tau \setminus (\sigma \cup (\operatorname{Lk} \sigma))$ then by (A.18), $\sigma(y) \in \operatorname{Bd} \sigma$, so by (A.25), we have $g(y) = y$. If $y \in \tau \cap (\operatorname{Lk} \sigma)$ then, by (A.25) again, $g(y) = y$. Therefore, by (A.34), we have $\mathcal{S}' \cap (\tau \setminus \sigma) = \mathcal{S} \cap (\tau \setminus \sigma)$ and $\Phi' = \Phi$ on $\tau \setminus (\sigma \cup \mathcal{S}')$.

Finally, suppose $\tau \nsubseteq \overline{\operatorname{St}}\, \sigma$ (i.e., τ is not a face of any $\rho \in P$ with $\sigma \subset \rho$). Write

$$\mathcal{S}' \cap (\tau \setminus \sigma) = \left[\mathcal{S}' \cap \Big(\big[\tau \cap (\overline{\operatorname{St}}\, \sigma)\big] \setminus \sigma\Big)\right] \cup \Big[(\mathcal{S}' \cap \tau) \setminus (\overline{\operatorname{St}}\, \sigma)\Big].$$

By (A.25), g is the identity map on $(\mathcal{S}' \cap \tau) \setminus (\overline{\operatorname{St}}\, \sigma)$. Moreover, by (A.26) and (A.28),

$$g : (\overline{\operatorname{St}}\, \sigma) \setminus \big[\mathcal{C} \cap (\operatorname{Bd} \sigma)\big] \to \overline{\operatorname{St}}\, \sigma.$$

Hence, by (2.38), we have $(\mathcal{S}' \cap \tau) \setminus (\overline{\operatorname{St}}\, \sigma) = (\mathcal{S} \cap \tau) \setminus (\overline{\operatorname{St}}\, \sigma)$ and $\Phi' = \Phi$ on $\tau \setminus \big[\mathcal{S} \cup (\overline{\operatorname{St}}\, \sigma)\big] = \big[\tau \setminus (\overline{\operatorname{St}}\, \sigma)\big] \setminus \big[(\mathcal{S} \cap \tau) \setminus (\overline{\operatorname{St}}\, \sigma)\big]$.

Furthermore, by (B.5), $\tau \cap (\overline{\operatorname{St}}\, \sigma)$ is the union of simplices in $\overline{\operatorname{St}}\, \sigma$ that are also faces of τ. Since $\tau \nsubseteq \overline{\operatorname{St}}\, \sigma$, none of these simplices equal σ (otherwise τ would have σ as a face and so would lie in $\overline{\operatorname{St}}\, \sigma$) and the dimension of each of these simplices is less than n. We have already proved that point (f) of the lemma applies to such simplices. Summing up,

$$\mathcal{S}' \cap (\tau \setminus \sigma) = \left[\mathcal{S} \cap \Big(\big[\tau \cap (\overline{\operatorname{St}}\, \sigma)\big] \setminus \sigma\Big)\right] \cup \Big[(\mathcal{S} \cap \tau) \setminus (\overline{\operatorname{St}}\, \sigma)\Big] = \mathcal{S} \cap (\tau \setminus \sigma)$$

and $\Phi' = \Phi$ on $\tau \setminus (\sigma \cup \mathcal{S})$. This proves point (f) of the lemma.

To prove point (g) first write

$$(\operatorname{Int} \tau) \cap \mathcal{S}' = \big[(\operatorname{Int} \tau) \cap \mathcal{S}' \cap (\operatorname{Int} \sigma)\big] \cup \big[(\operatorname{Int} \tau) \cap \mathcal{S}' \cap (\operatorname{Bd} \sigma)\big] \cup \big[(\operatorname{Int} \tau) \cap \mathcal{S}' \setminus \sigma\big]. \tag{A.35}$$

Since $\dim \tau < n := \dim \sigma$ we have $\tau \neq \sigma$ so, by (B.5'),

$$(\operatorname{Int} \tau) \cap \mathcal{S}' \cap (\operatorname{Int} \sigma) = \varnothing = (\operatorname{Int} \tau) \cap \mathcal{S} \cap (\operatorname{Int} \sigma). \tag{A.36}$$

In particular,

$$(\operatorname{Int} \tau) \cap \mathcal{S} \cap \sigma = (\operatorname{Int} \tau) \cap \mathcal{S} \cap (\operatorname{Bd} \sigma). \tag{A.37}$$

By (A.29) and (2.6) and (2.11),

$$\begin{aligned}(\operatorname{Int} \tau) \cap \mathcal{S}' \cap (\operatorname{Bd} \sigma) &= (\operatorname{Int} \tau) \cap \big[\mathcal{C}_z \cap (\operatorname{Bd} \sigma)\big] \\ &= (\operatorname{Int} \tau) \cap \Big(\big[\mathcal{S} \cap (\operatorname{Bd} \sigma)\big] \cup \big[\mathcal{C}_z \cap (\operatorname{Bd} \sigma)\big]\Big) \\ &= (\operatorname{Int} \tau) \cap \Big(\big[\mathcal{S} \cap (\operatorname{Bd} \sigma)\big] \cup \bar{h}_{z,\sigma}(\mathcal{S} \cap \sigma)\Big).\end{aligned} \tag{A.38}$$

Finally, by point (f) of the lemma, we have

$$(\mathrm{Int}\,\tau) \cap (\mathcal{S}' \setminus \sigma) = (\mathrm{Int}\,\tau) \cap (\mathcal{S} \setminus \sigma). \tag{A.39}$$

Point (g) of the lemma follows from applying (A.36), (A.38), (A.39), and (A.37) to (A.35).

To prove point (h) first note that if $\rho \in P$ but $\rho \not\subset \overline{\mathrm{St}}\,\sigma$ (i.e., ρ is not a face of any simplex in P having σ as a face; in particular ρ is not a face of σ; ρ cannot be a proper face of σ anyway because $\dim \rho \geq n$) then $\mathcal{S}' \cap (\mathrm{Int}\,\rho) = \mathcal{S} \cap (\mathrm{Int}\,\rho)$ by (A.31) and (A.24). Suppose $\rho \in P$ and $\rho \subset \overline{\mathrm{St}}\,\sigma$. If $\rho \subset \mathrm{Lk}\,\sigma$ then by point (e) of the lemma, we have $\mathcal{S}' \cap (\mathrm{Int}\,\tau) = \mathcal{S} \cap (\mathrm{Int}\,\tau)$. So suppose $\rho \subset \overline{\mathrm{St}}\,\sigma$ but $\rho \not\subset \mathrm{Lk}\,\sigma$. Thus, $\rho \cap \sigma \neq \varnothing$. If $\sigma(\mathrm{Int}\,\rho) \subset \mathrm{Bd}\,\sigma$ then by (A.31) again and (A.25), we have $\mathcal{S}' \cap (\mathrm{Int}\,\rho) = \mathcal{S} \cap (\mathrm{Int}\,\rho)$. Suppose $\sigma(\mathrm{Int}\,\rho) \cap (\mathrm{Int}\,\sigma) \neq \varnothing$. Then by (A.18), σ is a face of ρ. If $\rho = \sigma$ then $\mathcal{S}' \cap (\mathrm{Int}\,\rho) = \varnothing$ by (A.29) and point (h) of the lemma holds trivially. So suppose σ is a proper face of ρ. Then $(\mathrm{Int}\,\rho) \cap \sigma = \varnothing$. It follows from (A.31)

$$\mathcal{S}' \cap (\mathrm{Int}\,\rho) = g^{-1}(\mathcal{S} \cap (\mathrm{Int}\,\rho)).$$

By point (b) of the lemma, g^{-1} is locally Lipschitz on $\mathrm{Int}\,\rho$. Point (h) then follows from lemma C.2 and (C.4) in appendix C.

Next, we prove point (i). Suppose $s \geq 0$, $\tau \in P$, and $\mathcal{H}^s(\mathcal{S}' \cap (\mathrm{Int}\,\tau)) > 0$. We show that either $\mathcal{H}^s(\mathcal{S} \cap (\mathrm{Int}\,\tau)) > 0$ or τ is a proper face of σ and $\mathcal{H}^s(\mathcal{S} \cap (\mathrm{Int}\,\sigma)) > 0$. If $\dim \tau \geq n$ the claim is immediate from point (h) of the lemma. So suppose $\tau \in P$ with $\dim \tau < n$ and suppose $\mathcal{H}^s(\mathcal{S}' \cap (\mathrm{Int}\,\tau)) > 0$. First, suppose τ is not a face of σ. Then, by (B.6) in appendix B, we have $\mathrm{Int}\,\tau \cap \sigma = \varnothing$. Hence, $\mathrm{Int}\,\tau \subset \tau \setminus \sigma$. Thus, by point (f) of the lemma,

$$\mathcal{S}' \cap (\mathrm{Int}\,\tau) = \big[\mathcal{S}' \cap (\tau \setminus \sigma)\big] \cap (\mathrm{Int}\,\tau) = \big[\mathcal{S} \cap (\tau \setminus \sigma)\big] \cap (\mathrm{Int}\,\tau) = \mathcal{S} \cap (\mathrm{Int}\,\tau). \tag{A.40}$$

Hence, $\mathcal{H}^s(\mathcal{S} \cap (\mathrm{Int}\,\tau)) > 0$.

Suppose τ is a face of σ. Then it is a proper face, since $\dim \tau < n$. I.e., $\tau \subset \mathrm{Bd}\,\sigma$. By point (a) of lemma 2.1, (2.11) and (2.9), we have

$$\begin{aligned}\mathcal{H}^s\big(\mathcal{S}' \cap (\mathrm{Int}\,\tau)\big) &\leq \mathcal{H}^s\big[\mathcal{S} \cap (\mathrm{Int}\,\tau)\big] + \mathcal{H}^s\Big(\bar{h}\big[\mathcal{S} \cap (\mathrm{Int}\,\sigma)\big] \cap (\mathrm{Int}\,\tau)\Big) \\ &\leq \mathcal{H}^s\big[\mathcal{S} \cap (\mathrm{Int}\,\tau)\big] + \mathcal{H}^s\Big(\bar{h}\big[\mathcal{S} \cap (\mathrm{Int}\,\sigma)\big]\Big).\end{aligned}$$

Thus, if $\mathcal{H}^s\big[\mathcal{S} \cap (\mathrm{Int}\,\tau)\big] = 0$, we must have $\mathcal{H}^s\Big(\bar{h}\big[\mathcal{S} \cap (\mathrm{Int}\,\sigma)\big]\Big) > 0$. Therefore, by lemma A.2 part (7), and lemma C.2 we have $\mathcal{H}^s\big[\mathcal{S} \cap (\mathrm{Int}\,\sigma)\big] > 0$. This completes the proof of point (i).

Next, point (j). Write $\mathcal{S}' = (\mathcal{S}' \cap \sigma) \cup (\mathcal{S}' \setminus \sigma)$ and $\mathcal{S} = (\mathcal{S} \cap \sigma) \cup (\mathcal{S} \setminus \sigma)$. By (2.39), $\dim(\mathcal{S}' \cap \sigma) \leq \dim(\mathcal{S} \cap \sigma)$. So, by (C.5) in appendix C, it suffices to show $\dim(\mathcal{S}' \setminus \sigma) \leq \dim(\mathcal{S} \setminus \sigma)$. Note that, by (B.6) and (B.7),

$$\mathcal{S}' \setminus \sigma = \bigcup_{\tau \in P,\, \tau \not\subset \sigma} \mathcal{S}' \cap (\mathrm{Int}\,\tau).$$

A similar formula holds for $\mathcal{S}$. Thus, by (C.5) again, it suffices to show that

$$\dim\big[\mathcal{S}' \cap (\mathrm{Int}\,\tau)\big] \leq \dim\big[\mathcal{S} \cap (\mathrm{Int}\,\tau)\big] \tag{A.41}$$

for every $\tau \in P$ s.t. $\tau \not\subset \sigma$. Let $\tau \in P$ and suppose $\tau \not\subset \sigma$. If $\dim \tau \geq n = \dim \sigma$ then (A.41) holds by point (h) of the lemma. Suppose that $\dim \tau < n = \dim \sigma$. Then τ is not a face of σ so (A.40) implies that $\dim\big[\mathcal{S}' \cap (\mathrm{Int}\,\tau)\big] = \dim\big[\mathcal{S} \cap (\mathrm{Int}\,\tau)\big]$. This shows that $\dim \mathcal{S}' \leq \dim \mathcal{S}$. The same argument shows $\dim\big(\mathcal{S}' \cap |Q|\big) \leq \dim\big(\mathcal{S} \cap |Q|\big)$.

We prove point (k) of the lemma. First, note that by (B.7) it suffices to show

$$\text{For every } \rho \in P \text{ we have } \Phi'\big((\operatorname{Int}\rho) \setminus \mathcal{S}'\big) \subset \Phi\big((\operatorname{Int}\rho) \setminus \mathcal{S}\big). \tag{A.42}$$

Let $\rho \in P$. First, suppose $\rho \nsubseteq \sigma$ and let $y \in (\operatorname{Int}\rho) \setminus \mathcal{S}'$. Then $y \notin \sigma$ by (B.6). Hence, by (A.28) and (2.38) $g(y) \notin \mathcal{S}$. Moreover, by point (b), $g(y) \in \operatorname{Int}\rho$. Hence,

$$\Phi'(y) = \Phi\big[g(y)\big] \in \Phi\big[(\operatorname{Int}\rho) \setminus \mathcal{S}\big].$$

Now suppose $\rho \subset \sigma$ and let $y \in (\operatorname{Int}\rho) \setminus \mathcal{S}'$. First, suppose ρ is a proper face of σ. Then by point (a) of the lemma, $y \in (\operatorname{Int}\rho) \setminus \mathcal{S}$ and, by (A.15), (A.25), and (2.37), $\Phi'(y) = \Phi(y)$. (A.42) follows in this case. Suppose $\rho = \sigma$. Then, by point (a) of the lemma, we have $(\operatorname{Int}\rho) \cap \mathcal{S}' = \varnothing$. If $y \in \mathcal{C}$, then by (A.24) and lemma A.2 (part 1), $g(y) = z \notin \mathcal{C}$. *A fortiori*, by (2.6), $g(y) \in (\operatorname{Int}\rho) \setminus \mathcal{S}$. Suppose $y \in \sigma \setminus \mathcal{C}$. Then by (A.24) and lemma A.2 (part 5), $g(y) \in \sigma \setminus \mathcal{C}$ and so, again, by (2.6), $g(y) \in \sigma \setminus \mathcal{S}$. This concludes the proof of point (k) of the lemma.

This concludes the proof of lemma 2.1. □

Proof of lemma A.2. Proof of part (1) Suppose $y \in \mathcal{C} \setminus (\operatorname{Bd}\sigma)$. By (2.13), $\bar{h}(y) \in \mathcal{C}$ so $\bar{\Delta}(y) = 0$ and therefore, $f(y, 0)$ defined by (2.33) is 1.

Proof of part (2) Easy consequence of (2.10) and (2.31).

Proof of part (3) Suppose $s(y,t) \in \operatorname{Bd}\sigma$. Then by definition of $s(y,t)$, either $y \in \operatorname{Bd}\sigma$ or $b(y) < 1$ and $f(y,t) = 0$. But by definitions of f and k, $b(y) < 1$ and $f(y,t) = 0$ are mutually contradictory. Thus, $y \in \operatorname{Bd}\sigma$.

Proof of part (4) Use property (2) .

Proof of part (5) Suppose $s(y,t) \in \mathcal{C}$. If $s(y,t) \in \operatorname{Bd}\sigma$ then, by property (3), $y = s(y,t) \in \mathcal{C}$. What if $s(y,t) \notin \operatorname{Bd}\sigma$? Then $y \notin \operatorname{Bd}\sigma$, by property (3) again, so $b(y) \in [0,1)$. Suppose $y \notin \mathcal{C}$. By (2.6), $y \neq z$ so by (2.13) $\bar{h}(y) \notin \mathcal{C}$. Therefore, since $\mathcal{C}$ is compact ((2.6) again), $\bar{\Delta}(y) > 0$. Hence,

$$k\big[b(y), \bar{\Delta}(y), t\big] < 1.$$

I.e., $s(y,t) \neq z$. Hence, by property (4), $\bar{h}\big[s(y,t)\big] = \bar{h}(y) \notin \mathcal{C}$. But by (2.13), this means $s(y,t) \notin \mathcal{C}$, contradiction. It follows that $y \in \mathcal{C}$.

Next, suppose $y \in \mathcal{C}$. If $y = z$, then by property (2), we have $s(y,t) = z \in \mathcal{C}$. Assume $s(y,t) \neq z$. Then by property (4), $y \neq z$ and $\bar{h}\big[s(y,t)\big] = \bar{h}(y) \in \mathcal{C}$ so by (2.13) again $s(y,t) \in \mathcal{C}$.

Proof of part (6) (A.10) is only interesting when

$$\bar{h}(y_2) \neq \bar{h}(y_1) \text{ and either } y_1 \neq z \text{ or } y_2 \neq z \text{ or both.} \tag{A.43}$$

Assume (A.43). Suppose $(y_1 - z) \cdot (y_2 - z) \leq 0$.

$$|y_2 - y_1| = \sqrt{|y_2 - z|^2 - 2(y_1 - z) \cdot (y_2 - z) + |y_1 - z|^2} \geq |y_i - z|, \quad i = 1, 2.$$

Thus,

$$|y_i - z||\bar{h}(y_2) - \bar{h}(y_1)| \leq diam(\sigma)|y_2 - y_1|, \text{ if } (y_1 - z) \cdot (y_2 - z) \leq 0. \tag{A.44}$$

I.e., (A.10) holds if $(y_1 - z) \cdot (y_2 - z) \leq 0$.

So suppose

$$(y_1 - z) \cdot (y_2 - z) > 0. \tag{A.45}$$

In particular, $y_1 \neq z$ and $y_2 \neq z$. Write $h_i := \bar{h}(y_i)$ and note that, since $y_i \neq z$, we have $b(y_i) > 0$, by (2.10) ($i = 1, 2$). Therefore, by (2.8) and (A.45)

$$(h_1 - z) \cdot (h_2 - z) > 0 \tag{A.46}$$

and all the points in (A.10) lie on the triangle $h_1 z h_2$.

WLOG

$$|h_1 - z| \geq |h_2 - z|. \tag{A.47}$$

Claim: Under (A.45) the angle, ω, between $z - h_1$ and $h_2 - h_1$ is bounded away from 0 by a value depending only on σ and $z \in \operatorname{Int} \sigma$. It suffices to treat the case $0 \leq \omega < \pi/2$. Let w lie on the line (unique because of (A.43)) passing through h_1 and h_2. Thus, for some $s \in \mathbb{R}$, we have $w - h_1 = s(h_2 - h_1)$. Since $0 \leq \omega < \pi/2$, we have $(h_2 - h_1) \cdot (h_1 - z) \neq 0$. Therefore, we may pick $s \in \mathbb{R}$ s.t. $w - z \perp h_1 - z$. Hence,

$$\begin{aligned} 0 - |h_1 - z|^2 &= (h_1 - z) \cdot \big[(w - z) - (h_1 - z)\big] \\ &= (h_1 - z) \cdot s\big[(h_2 - z) - (h_1 - z)\big] = s(h_1 - z) \cdot (h_2 - z) - s|h_1 - z|^2. \end{aligned}$$

Thus,

$$(s - 1)|h_1 - z|^2 = s(h_1 - z) \cdot (h_2 - z).$$

But $(h_1 - z) \cdot (h_2 - z) > 0$, by (A.46). Therefore, $s < 0$ or $s > 1$. If $s < 0$ then by Schwarz's inequality (Stoll and Wong [SW68, Theorem 3.1, p. 79]) and (A.47)

$$(s - 1)|h_1 - z|^2 = s(h_1 - z) \cdot (h_2 - z) \geq s|h_1 - z||h_2 - z| \geq s|h_1 - z|^2.$$

I.e., $-|h_1 - z|^2 \geq 0$. This is impossible since $h_1 \in \operatorname{Bd} \sigma$ while $z \in \operatorname{Int} \sigma$. Therefore, $s > 1$. Now,

$$s^{-1} w + (1 - s^{-1}) h_1 = h_2. \tag{A.48}$$

Let $\Pi \subset \mathbb{R}^N$ be the smallest (affine) plane containing σ. Thus, $\dim \Pi = n$ and $h_1, h_2 \in \Pi$. Since $w = (1 - s)h_1 + s h_2$, we also have $w \in \Pi$. Since $z \in \operatorname{Int} \sigma$, $r = r(z) := dist(z, \operatorname{Bd} \sigma) > 0$. The open ball B, of radius r centered at z satisfies $B \cap \Pi \subset \operatorname{Int} \sigma$. Suppose $\tan \omega < \epsilon = \epsilon(z) := r(z)/diam(\sigma) > 0$. (So $\epsilon(z)$ is continuous in z.) Now, $|h_1 - z| \leq diam(\sigma)$ and

$$\frac{r}{diam(\sigma)} > \tan \omega = \frac{|w - z|}{|h_1 - z|} \geq \frac{|w - z|}{diam(\sigma)}.$$

Thus, we must have $w \in B \cap \Pi \subset \operatorname{Int} \sigma$. (We also observe that $0 < \tan \omega < +\infty$ so $0 < \omega < \pi/2$.) Hence, from (A.48), we see that $h_2 \in \operatorname{Bd} \sigma$ lies on the open line segment joining $w \in \operatorname{Int} \sigma$ and $h_1 \in \sigma$. It easily follows that $h_2 \in \operatorname{Int} \sigma$, a contradiction. Therefore, $\omega \geq \omega_0 = \omega_0(z) := \arctan \epsilon(z)$. This proves the claim that ω is bounded away from 0. Note $\sin \omega \geq \sin \omega_0 = \epsilon / \sqrt{1 + \epsilon^2}$.

Let α be the angle between $h_1 - z$ and $h_2 - z$. By (A.46), $0 \leq \alpha < \pi/2$. Let $K_1'' = K_1''(z) := \frac{diam(\sigma)\sqrt{1+\epsilon(z)^2}}{\epsilon(z)}$. So $K_1''(z)$ is continuous in z. By the Law of Sines,

$$\frac{|h_2 - h_1|}{\sin \alpha} = \frac{|h_2 - z|}{\sin \omega} \leq K_1''. \tag{A.49}$$

Now, by (A.45), $y_1 \neq z$ and $y_2 \neq z$. Thus, α is the angle between $y_1 - z$ and $y_2 - z$. Therefore, by (A.49),

$$\begin{aligned}
|h_2 - h_1| &\leq K_1'' \sqrt{1 - \cos^2 \alpha} \\
&= K_1'' \sqrt{(1 + \cos\alpha)(1 - \cos\alpha)} \\
&\leq \sqrt{2} K_1'' \sqrt{1 - \cos\alpha} \\
&= \sqrt{2} K_1'' \sqrt{1 - \frac{(y_1 - z) \cdot (y_2 - z)}{|y_1 - z||y_2 - z|}} \qquad \text{(A.50)} \\
&= \sqrt{2} K_1'' \sqrt{1 + \frac{|y_1 - y_2|^2 - |y_1 - z|^2 - |y_2 - z|^2}{2|y_1 - z||y_2 - z|}} \\
&= \sqrt{2} K_1'' \sqrt{1 + \frac{|y_1 - y_2|^2}{2|y_1 - z||y_2 - z|} - \frac{|y_1 - z|}{2|y_2 - z|} - \frac{|y_2 - z|}{2|y_1 - z|}}.
\end{aligned}$$

Now, for $a, b > 0$ we have $\frac{a}{2b} + \frac{b}{2a} \geq 1$. Applying this general inequality to (A.50) we get

$$|h_2 - h_1| \leq \sqrt{2} K_1'' \frac{1}{\sqrt{2|y_1 - z||y_2 - z|}} |y_1 - y_2|.$$

Multiplying the extreme members of the preceding by $\min_{i=1,2} |y_i - z|$ we get

$$\begin{aligned}
\text{(A.51)} \quad \Big(\min_{i=1,2} |y_i - z|\Big) \big|\bar{h}(y_2) - \bar{h}(y_1)\big| &= |h_2 - h_1| \Big(\min_{i=1,2} |y_i - z|\Big) \\
&\leq \sqrt{2} K_1'' \frac{\min_{i=1,2} |y_i - z|}{\sqrt{2|y_1 - z||y_2 - z|}} |y_1 - y_2| \leq K_1'' |y_1 - y_2|.
\end{aligned}$$

(By definition of K_1'', we have $K_1'' \geq diam(\sigma)$. Therefore, by (A.44) , the same K_1'' works if $(y_1 - z) \cdot (y_2 - z) \leq 0$.) Let $K'' = K''(z) = diam(\sigma) + K_1''(z)$. Then $K''(z)$ depends only on σ and z and is continuous in z.

Suppose $|y_1 - z| \leq |y_2 - z|$. Then, by (A.51), (A.10) holds with $i = 1$. But

$$\begin{aligned}
|y_2 - z||\bar{h}(y_2) - \bar{h}(y_1)| &\leq |y_2 - y_1||\bar{h}(y_2) - \bar{h}(y_1)| + |y_1 - z||\bar{h}(y_2) - \bar{h}(y_1)| \\
&\leq \big(diam(\sigma) + K_1''\big)|y_1 - y_2| \\
&= K''|y_2 - y_1|.
\end{aligned}$$

Similarly if $|y_2 - z| \leq |y_1 - z|$.

Proof of part (7) As previously observed, it follows from lemma B.2 that $b(y)$ is Lipschitz in $y \in \sigma$. We give another proof here. Let $y_1, y_2 \in \sigma$. WLOG $|y_2 - z| \leq |y_1 - z|$. By (2.8)

$$y_2 - y_1 = \big[b(y_2) - b(y_1)\big]\big[\bar{h}(y_1) - z\big] + b(y_2)\big[\bar{h}(y_2) - \bar{h}(y_1)\big]. \qquad \text{(A.52)}$$

Now, $\bar{h}(y_2) \in \operatorname{Bd}\sigma$ (even if $y_2 = z$) and $dist(z, \operatorname{Bd}\sigma) > 0$. Moreover, by (2.8),

$$b(y_2) = |y_2 - z| / \big|\bar{h}(y_2) - z\big|, \qquad \text{(A.53)}$$

even if $y_2 = z$. Thus, by (A.52) and part (6),

$$\begin{aligned} dist(z, \operatorname{Bd}\sigma)\big|b(y_2) - b(y_1)\big| &\le \big|b(y_2) - b(y_1)\big|\big|\bar{h}(y_1) - z\big| \\ &\le |y_2 - y_1| + b(y_2)\big|\bar{h}(y_2) - \bar{h}(y_1)\big| \\ &= |y_2 - y_1| + \frac{|y_2 - z|}{|\bar{h}(y_2) - z|}\big|\bar{h}(y_2) - \bar{h}(y_1)\big| \\ &\le |y_2 - y_1| + dist(z, \operatorname{Bd}\sigma)^{-1}|y_2 - z|\big|\bar{h}(y_2) - \bar{h}(y_1)\big| \\ &\le |y_2 - y_1| + dist(z, \operatorname{Bd}\sigma)^{-1}K''|y_2 - y_1|. \end{aligned}$$

Thus, (A.11) holds.

If $y \in \sigma \setminus \{z\}$, we have, by (2.10), that $b(y) > 0$ and, by (2.8),

$$\bar{h}(y) = b(y)^{-1}(y - z) + z.$$

It follows from (A.11) and (C.8) that $\bar{h} : \sigma \setminus \{z\} \to \operatorname{Bd}\sigma$ is locally Lipschitz on $\sigma \setminus \{z\}$. Similarly, since b is bounded away from 0 on $\mathcal{A}$ (because $\mathcal{A}$ is compact and disjoint from z; (2.3), (2.4), and (2.10)), $\bar{h}$ is actually Lipschitz on $\mathcal{A}$.

Proof of part (8) We wish to show that f is locally Lipschitz on B_z. Now, by property (7), $\bar{h}(y)$ is locally Lipschitz in $y \in \sigma \setminus \{z\}$ and . Therefore, $\bar{\Delta}(y)$ is locally Lipschitz in $y \in \sigma \setminus \{z\}$. Moreover, $(y, t) \in B_z$ implies $\big(b(y), \bar{\Delta}(y), t\big) \in T$ defined in (A.7). Hence, since $b(y)$ is Lipschitz in σ it follows from (C.8) and lemma A.1 that f is locally Lipschitz on $\big\{(y, t) \in B_z : y \neq z\big\}$. Therefore, to prove part (8), it suffices to show that f is locally Lipschitz in $(\operatorname{Int}\sigma) \times [0, 1]$.

Let $y_i \in \operatorname{Int}\sigma$, $t_i \in [0, 1]$ and write $b_i = b(y_i)$, $\Delta_i = \bar{\Delta}(y_i)$, and $\xi_i = \big(b_i, \Delta_i, t_i\big)$ $(i = 1, 2)$. Since $y_1, y_2 \in \operatorname{Int}\sigma$ we have, by (2.10), $b_1, b_2 < 1$. In fact, we may assume that for some $\epsilon > 0$, we have $b_1, b_2 < 1 - \epsilon$.

Now, $\xi_1, \xi_2 \in T$ defined (A.7) and, lemma A.1 tells us, T is convex. Therefore, by the multivariate Mean Value Theorem (Apostol [Apo57, 6–17, p. 117]) and (A.6) there exists $\xi = (b, \Delta, t)$ in the line segment joining ξ_1 and ξ_2 s.t.

$$\begin{aligned} \big|f(y_2, t_2) - f(y_1, t_1)\big| &= \big|(\xi_2 - \xi_1) \cdot \nabla k(\xi)\big| \\ &= \left|\frac{(b_2 - b_1)(\Delta + t)}{(1 - b)^2} + b\frac{\Delta_2 - \Delta_1}{1 - b} + b\frac{t_2 - t_1}{1 - b}\right| \exp\left\{-\frac{b}{1 - b}(\Delta + t)\right\}. \end{aligned}$$

Now, $b_1, b_2 < 1 - \epsilon$ implies $b < 1 - \epsilon$. Therefore, from (A.11), we know that there exists $K < \infty$, depending only on ϵ and σ, s.t.

$$\left|\frac{(b_2 - b_1)(\Delta + t)}{(1 - b)^2}\right| \le K\big|(y_2, t_2) - (y_1, t_1)\big| \text{ and } \left|b\frac{t_2 - t_1}{1 - b}\right| \le K\big|(y_2, t_2) - (y_1, t_1)\big|.$$

So it suffices to show that, at the possible cost of increasing K, we have

$$\left|b\frac{\Delta_2 - \Delta_1}{1 - b}\right| \le K\big|(y_2, t_2) - (y_1, t_1)\big|. \tag{A.54}$$

Let $h_i = \bar{h}(y_i)$. By compactness of $\mathcal{C}$ ((2.6)), there exists $w_i \in \mathcal{C}$ s.t. $\Delta_i = |h_i - w_i|$ $(i = 1, 2)$. WLOG $\Delta_2 \geq \Delta_1$. Thus,

$$\begin{aligned} |\Delta_2 - \Delta_1| &= |h_2 - w_2| - |h_1 - w_1| \\ &\leq |h_2 - w_1| - |h_1 - w_1| \\ &\leq |h_2 - w_1 - h_1 + w_1| \\ &= |h_2 - h_1|. \end{aligned}$$

Thus, as in (A.53),

$$\begin{aligned} b|\Delta_2 - \Delta_1| &\leq b|h_2 - h_1| \\ &\leq (b_1 + b_2)|h_2 - h_1| \\ &= \left(\frac{|y_1 - z|}{|h_1 - z|} + \frac{|y_2 - z|}{|h_2 - z|} \right) |h_2 - h_1| \\ &\leq \frac{1}{dist(z, \operatorname{Bd} \sigma)} \big(|y_1 - z| + |y_2 - z| \big) |h_2 - h_1|. \end{aligned}$$

Now apply (A.10).

Proof of part (9) By part (8) of the lemma, f is locally Lipschitz on B_z. Moreover, by part (7) of the lemma, $\bar{h}$, and hence $\bar{\Delta}$, is locally Lipschitz on $\sigma \setminus \{z\}$. Therefore, by (C.8), $s(y, t)$ is locally Lipschitz on $B_z \setminus \big(\{z\} \times [0, 1]\big)$.

It remains to show that s is locally Lipschitz in $U \times [0, 1]$ for some neighborhood, U, of z. By part (8) of the lemma again and compactness, there exists a neighborhood, $U \subset \operatorname{Int} \sigma$, of z, s.t. f is actually Lipschitz on $U \times [0, 1]$, with Lipschitz constant $M < \infty$, say. We wish to show that there exists $K < \infty$ s.t.

$$\big|s(y_2, t_2) - s(y_1, t_1)\big| \leq K \big|(y_2, t_2) - (y_1, t_1)\big|, \quad \text{for } y_i \in U \text{ and } t_i \in [0, 1] \ (i = 1, 2). \tag{A.55}$$

Since $U \subset \operatorname{Int} \sigma$, by (2.10), $b(y_i) < 1$ and we have $(y_i, t_i) \in B_z$ $(i = 1, 2)$. Therefore, by part (8) of the lemma,

$$\begin{aligned} \big|s(y_2, t_2) - s(y_1, t_1)\big| &\leq \Big| \big(1 - f(y_2, t_2)\big) \bar{h}(y_2) - \big(1 - f(y_1, t_1)\big) \bar{h}(y_1) \Big| \\ &\qquad + \big|f(y_2, t_2) - f(y_1, t_1)\big| |z| \\ &\leq \Big| \big(1 - f(y_2, t_2)\big) \bar{h}(y_2) - \big(1 - f(y_1, t_1)\big) \bar{h}(y_1) \Big| \\ &\qquad + M|z| \big|(y_2, t_2) - (y_1, t_1)\big| \\ &\leq \Big| \big(1 - f(y_2, t_2)\big) \bar{h}(y_2) - \big(1 - f(y_1, t_1)\big) \bar{h}(y_1) \Big| \\ &\qquad + MD \big|(y_2, t_2) - (y_1, t_1)\big|, \end{aligned} \tag{A.56}$$

where $D < \infty$ is an upper bound on the norm of the points in σ. Now

$$\begin{aligned}(A.57)\quad &\Big|\big(1 - f(y_2,t_2)\big)\bar{h}(y_2) - \big(1 - f(y_1,t_1)\big)\bar{h}(y_1)\Big| \\ &\leq \big|1 - f(y_2,t_2)\big||\bar{h}(y_2) - \bar{h}(y_1)| \\ &\qquad + \big|f(y_1,t_1) - f(y_2,t_2)\big||\bar{h}(y_1)| \\ &\leq \big|1 - f(y_2,t_2)\big||\bar{h}(y_2) - \bar{h}(y_1)| + M\big|(y_2,t_2) - (y_1,t_1)\big||\bar{h}(y_1)| \\ &\leq \big|1 - f(y_2,t_2)\big||\bar{h}(y_2) - \bar{h}(y_1)| + MD\big|(y_2,t_2) - (y_1,t_1)\big|.\end{aligned}$$

Thus, to prove (A.55) it suffices to find $K' < \infty$ s.t.

$$\big|\bar{h}(y_2) - \bar{h}(y_1)\big|\big|1 - f(y_2,t_2)\big| \leq K'|y_2 - y_1|. \tag{A.58}$$

Now b is bounded away from 1 on U so by lemma A.1, $\frac{\partial}{\partial\beta}k\big(\beta, \bar{\Delta}(y_2), t_2\big)$ is bounded above for $\beta = b(y_2)$, $y_2 \in U$. Therefore, by (2.33), the mean value theorem, and (2.8),

$$\begin{aligned}\big|1 - f(y_2,t_2)\big| &= \big|k\big[0, \bar{\Delta}(y_2), t_2\big] - k\big[b(y_2), \bar{\Delta}(y_2), t_2\big]\big| \\ &\leq M'b(y_2) \leq M'\frac{|y_2 - z|}{dist(z, \operatorname{Bd}\sigma)},\end{aligned} \tag{A.59}$$

for some $M' < \infty$ valid throughout U. Substituting this into (A.58), (A.55) follows from part (6) of the lemma. □

Proof of lemma A.3. Let k be the function defined in (2.32) So

$$k(0,\delta,t) = 1, \text{ for every } \delta, t \in \mathbb{R}. \tag{A.60}$$

From lemma A.1, we have

$$k(\beta,\delta,t) \to 0 \text{ as } \beta \uparrow 1 \text{ for any } \delta \geq 0 \text{ and } t > 0. \tag{A.61}$$

From (A.6), we observe that

$$\frac{\partial}{\partial\beta}k(\beta,\delta,t) < 0, \quad \beta \in [0,1),\ \delta \geq 0,\ t > 0.$$

Hence,

$$\begin{aligned}(A.62)\quad &\text{For any } t > 0 \text{ and } \delta \geq 0, \\ &\qquad \text{the function } \beta \mapsto k(\beta,\delta,t) \text{ is strictly decreasing in } \beta \in [0,1).\end{aligned}$$

Combining (A.62) with (A.60) and (A.61) we see that

$$\text{The map } \beta \mapsto k(\beta,\delta,t) \text{ maps } [0,1) \text{ onto } (0,1] \text{ for any } \delta \geq 0 \text{ and } t > 0. \tag{A.63}$$

Notice that, by definition of k ((2.32)),

$$k(\beta,\delta,t) \leq \exp\left\{-\frac{\beta\epsilon}{1-\beta}\right\}, \quad \text{if } \beta \in [0,1),\ \delta \geq 0, \text{ and } 0 < \epsilon \leq t. \tag{A.64}$$

Hence, since $-\log k(\beta,\delta,t) > 0$,

$$\beta \leq \frac{-\log k(\beta,\delta,t)}{\epsilon - \log k(\beta,\delta,t)}, \quad \text{if } \beta \in [0,1),\ \delta \geq 0, \text{ and } 0 < \epsilon \leq t. \tag{A.65}$$

Note that the right hand side (RHS) of the preceding is strictly less than 1. From (A.5), we see

$$k(\beta, \delta, t) \leq (1-\beta)^{\epsilon}, \quad \text{if } \beta \in [0,1),\ \delta \geq 0,\ \text{and } 0 < \epsilon \leq t.$$

Thus,

$$\epsilon^{-1}\Big[1 - k(\beta, \delta, t)\Big] \geq \epsilon^{-1}\big[1 - (1-\beta)^{\epsilon}\big], \quad \text{if } \beta \in [0,1),\ \delta \geq 0,\ \text{and } 0 < \epsilon \leq t. \tag{A.66}$$

For $0 \leq \beta < 1$ and $0 < \epsilon \leq 1$, let $j(\beta) = 1 - (1-\beta)^{\epsilon} - \epsilon\beta$. Then $j(0) = 0$ and $j'(\beta) = (1-\beta)^{\epsilon-1} - \epsilon \geq 0$ for $0 \leq \beta < 1$ and $0 < \epsilon \leq 1$. Hence, $j(\beta) \geq 0$ for $0 \leq \beta < 1$ and $0 < \epsilon \leq 1$. Thus, from (A.66)

$$\epsilon^{-1}\Big[1 - k(\beta, \delta, t)\Big] \geq \beta, \quad \text{if } \beta \in [0,1),\ \delta \geq 0,\ \text{and } 0 < \epsilon \leq t \leq 1. \tag{A.67}$$

Claim:

$$\text{For any } t \in (0,1] \text{ we have } s\big[(\operatorname{Int}\sigma) \times \{t\}\big] = \operatorname{Int}\sigma. \tag{A.68}$$

By (A.9), we have $s\big[(\operatorname{Int}\sigma) \times \{t\}\big] \subset \sigma$. It follows from lemma A.2 (part 3) that $s\big[(\operatorname{Int}\sigma) \times \{t\}\big] \subset \operatorname{Int}\sigma$.

To prove the reverse inclusion, note that by lemma A.2(part 2), for every $t \in (0,1]$, $z \in s\big[(\operatorname{Int}\sigma) \times \{t\}\big]$. Let $w \in (\operatorname{Int}\sigma) \setminus \{z\}$ and $t \in (0,1]$. Then, by (2.10), $0 < b(w) < 1$ and by (2.8)

$$w = \bar{h}(w) + \big(1 - b(w)\big)\big[z - \bar{h}(w)\big].$$

By (A.63), we may pick $\beta \in [0,1)$ s.t. $k\big(\beta, \bar{\Delta}(w), t\big) = 1 - b(w)$. Since $1 - b(w) \in (0,1)$, by (A.60) and (2.32), $1 > \beta > 0$. Let $y = \bar{h}(w) + (1-\beta)\big[z - \bar{h}(w)\big] \in \operatorname{Int}\sigma$. Then $\bar{h}(y) = \bar{h}(w)$ (so $\bar{\Delta}(y) = \bar{\Delta}(w)$), $b(y) = \beta$, and by (A.8),

$$s(y,t) = \bar{h}(w) + \big(1 - b(w)\big)\big[z - \bar{h}(w)\big] = w.$$

This proves the claim (A.68).

Let $y \in \operatorname{Int}\sigma$ and $t \in (0,1]$. If $s(y,t) = z$ then lemma A.2(part 2) implies $y = z$ (since $t > 0$). Suppose $s(y,t) \neq z$. Then, by lemma A.2(part 4), $\bar{h}(y) = \bar{h}\big[s(y,t)\big]$ can be determined by $s(y,t)$. Hence, $\bar{\Delta}(y)$ and, by (A.8), $k\big(b(y), \bar{\Delta}(y), t\big)$ can be determined by $s(y,t)$. Therefore, by (A.62), $b(y)$ and, hence, y can be determined from $\hat{s}(y,t)$. To sum up:

$$\hat{s} \text{ is invertible on } (\operatorname{Int}\sigma) \times (0,1].$$

By lemma A.2 (part 7) and (2.10), if $B_{max} \in (0,1)$, then

$$W(B_{max}) := \big\{y \in \sigma : b(y) < B_{max}\big\}$$

is an open neighborhood of z whose closure lies in $\operatorname{Int}\sigma$. To prove that $\hat{s}$ has a locally Lipschitz inverse on $(\operatorname{Int}\sigma) \times (0,1]$, it suffices to show that its inverse is *Lipschitz* on $W(B_{max}) \times (\epsilon, 1]$ for arbitrary $B_{max} \in (0,1)$ and $\epsilon \in (0,1)$.

Let $B_{max} \in (0,1)$ and $\epsilon \in (0,1)$, let $y, y' \in \operatorname{Int}\sigma$, let $t, t' \in (\epsilon, 1]$ and suppose $s(y,t), s(y',t') \in W := W(B_{max})$. Write

$$x = \bar{h}\big(s(y,t)\big),\ B = b\big(s(y,t)\big) = 1 - k\big(b(y), \bar{\Delta}(y), t\big),\ x' = \bar{h}\big(s(y',t')\big),$$
$$\text{and } B' = b\big(s(y',t')\big) = 1 - k\big(b(y'), \bar{\Delta}(y'), t'\big).$$

So $s(y,t) = Bx + (1-B)z$ and $s(y',t') = B'x' + (1-B')z$. Since $s(y,t) \in W(B_{max})$, we have $B < B_{max}$. Hence,

$$k\big(b(y), \bar{\Delta}(y), t\big) > 1 - B_{max}.$$

Similarly for B'. Therefore, by (A.65) and (A.67), there exists $\beta_{max} \in (0,1)$, depending only on B_{max} and ϵ, s.t.

$$b(y) < \min\Big\{\beta_{max},\ \epsilon^{-1}B\Big\}. \tag{A.69}$$

Similarly for $b(y')$.

First, assume $s(y,t) = z$. Then, by (2.10), we have $b\big(s(y,t)\big) = 0$. Moreover, since $t > \epsilon > 0$, by lemma A.2 (part 2), $y = z$. Hence, WLOG $y' \neq y = z$. Then, by (2.8), (A.69), and (A.11) there exists $K_3 < \infty$ s.t.

$$\begin{aligned}
|y - y'| &= |y' - z| \\
&= b(y')\big|\bar{h}(y') - z\big| \\
&\leq \epsilon^{-1}B'\ diam(\sigma) \\
&= \epsilon^{-1}\ diam(\sigma)\Big|b\big(s(y',t')\big) - 0\Big| \\
&= \epsilon^{-1}\ diam(\sigma)\Big|b\big(s(y',t')\big) - b\big(s(y,t)\big)\Big| \\
&\leq K_3\epsilon^{-1}\ diam(\sigma)\big|s(y',t') - s(y,t)\big|.
\end{aligned}$$

This proves the lemma in the case where either $s(y,t) = z$ or $s(y',t') = z$.

Now assume $s(y,t), s(y',t') \in W \setminus \{z\}$. Then, by lemma A.2 (part 4),

$$x = \bar{h}\big(s(y,t)\big) = \bar{h}(y) \text{ and } x' = \bar{h}\big(s(y',t')\big) = \bar{h}(y'). \tag{A.70}$$

Claim: There exists $K_1 < \infty$ depending only on σ, z, B_{max}, and ϵ s.t.

$$b(y)|x' - x| \leq K_1\big|s(y',t') - s(y,t)\big|. \tag{A.71}$$

By (A.69), (2.10), and (A.11), there exists $K_2 < \infty$ and by lemma A.2 (part 6), there exists $K'' < \infty$ (K_2 and K'' depending only on σ and z) s.t.

$$\begin{aligned}
b(y)|x' - x| &\leq \epsilon^{-1}B|x' - x| \\
&= \epsilon^{-1}\Big|b\big(s(y,t)\big) - b(z)\Big|\Big|\bar{h}\big(s(y',t')\big) - \bar{h}\big(s(y,t)\big)\Big| \\
&\leq \epsilon^{-1}K_2\big|s(y,t) - z\big|\Big|\bar{h}\big(s(y',t')\big) - \bar{h}\big(s(y,t)\big)\Big| \\
&\leq \epsilon^{-1}K_2K''\big|s(y',t') - s(y,t)\big|.
\end{aligned}$$

This proves the claim (A.71).

Next, we *claim* that there exists $K_4 < \infty$ depending only on σ, z, B_{max}, and ϵ s.t.

$$\big|b(y') - b(y)\big| \leq K_4\Big(\big|s(y',t') - s(y,t)\big| + |t' - t|\Big). \tag{A.72}$$

WLOG $b(y) \geq b(y')$. Thus, since $t, t' \in (\epsilon, 1]$, if $(\tilde{\beta}, \tilde{\delta}, \tilde{t})$ is a point on the line joining $\big(b(y'), \bar{\Delta}(y'), t'\big)$ and $\big(b(y), \bar{\Delta}(y), t\big)$ we have $\tilde{\beta} \leq b(y)$ and $\tilde{t} > \epsilon$. By lemma A.1, the multivariate Mean Value Theorem (Apostol [Apo57, 6–17, p. 117]), and (A.69), we may choose

$(\tilde{\beta}, \tilde{\delta}, \tilde{t})$ s.t.

$$\begin{aligned}
&\Big|k\big(b(y'), \bar{\Delta}(y'), t'\big) - k\big(b(y), \bar{\Delta}(y), t\big)\Big| \\
&\qquad = \left| \frac{\tilde{\delta} + \tilde{t}}{(1-\tilde{\beta})^2}\big(b(y') - b(y)\big) + \frac{\tilde{\beta}}{1-\tilde{\beta}}\big(\bar{\Delta}(y') - \bar{\Delta}(y)\big) + \frac{\tilde{\beta}}{1-\tilde{\beta}}(t'-t) \right| \\
&\qquad \geq \frac{\tilde{\delta} + \tilde{t}}{(1-\tilde{\beta})^2}\big|b(y') - b(y)\big| - \frac{\tilde{\beta}}{1-\tilde{\beta}}\big|\bar{\Delta}(y') - \bar{\Delta}(y)\big| - \frac{\tilde{\beta}}{1-\tilde{\beta}}|t'-t| \\
&\qquad \geq \epsilon\big|b(y') - b(y)\big| - \frac{b(y)}{1-\beta_{max}}\big|\bar{\Delta}(y') - \bar{\Delta}(y)\big| - \frac{\beta_{max}}{1-\beta_{max}}|t'-t|.
\end{aligned}$$

Thus,

$$\begin{aligned}
\text{(A.73)} \quad \big|b(y') - b(y)\big| \leq \epsilon^{-1}&\Big|k\big(b(y'), \bar{\Delta}(y'), t'\big) - k\big(b(y), \bar{\Delta}(y), t\big)\Big| \\
&+ \frac{b(y)}{\epsilon(1-\beta_{max})}\big|\bar{\Delta}(y') - \bar{\Delta}(y)\big| + \frac{\beta_{max}}{\epsilon(1-\beta_{max})}|t'-t|.
\end{aligned}$$

Now, by (A.11) there exists $K'' < \infty$ s.t.,

$$\begin{aligned}
\text{(A.74)} \quad \Big|k\big(b(y'), \bar{\Delta}(y'), t'\big) - k\big(b(y), \bar{\Delta}(y), t\big)\Big| &= |B - B'| \\
&= \Big|b\big(s(y',t')\big) - b\big(s(y,t)\big)\Big| \leq K''\big|s(y',t') - s(y,t)\big|.
\end{aligned}$$

In addition, by definition of $\bar{\Delta}$ ((2.30)), example C.1, (A.70), and (A.71), we have

$$b(y)\big|\bar{\Delta}(y') - \bar{\Delta}(y)\big| \leq b(y)|x - x'| \leq K_1\big|s(y',t') - s(y,t)\big|.$$

Substituting this and (A.74) into (A.73) yields the claimed (A.72)

Finally, we show that there exists $K_5 < \infty$ depending only on σ, z, B_{max}, and ϵ s.t.

$$\text{(A.75)} \qquad |y' - y| \leq K_5\big|\hat{s}(y',t') - \hat{s}(y,t)\big|.$$

To see this, argue as follows using (2.8) and (A.70)

$$\begin{aligned}
|y' - y| &= \Big|\big(b(y') - b(y)\big)\big(\bar{h}(y') - z\big) + b(y)\big(\bar{h}(y') - \bar{h}(y)\big)\Big| \\
&\leq \big|b(y') - b(y)\big| diam(\sigma) + b(y)\big|\bar{h}(y') - \bar{h}(y)\big| \\
&= diam(\sigma)\big|b(y') - b(y)\big| + b(y)\big|x' - x\big|.
\end{aligned}$$

(A.75) now follows from (A.72) and (A.71). Since, trivially, $|t - t'| \leq \big|\hat{s}(y',t') - \hat{s}(y,t)\big|$ lemma A.3 follows. $\square$

Proof of lemma A.4. Let $x \in |P| \setminus \big[\mathcal{C} \cap (\operatorname{Bd}\sigma)\big]$ and let $\tau \in P$ satisfy $x \in \operatorname{Int}\tau$. (See (B.7).) We re-use an idea that we used in the proof of corollary 1.2. Write $\overline{\operatorname{St}}_x := \overline{\operatorname{St}}\,\tau$. Since $\overline{\operatorname{St}}_x$ is starlike w.r.t. x (appendix B), we have

$$\bar{V}_{t,x} := t(\overline{\operatorname{St}}_x - x) + x \subset \overline{\operatorname{St}}_x, \quad t \in (0,1).$$

Here, the vector operations are performed point-wise. Let

$$V_{t,x} := t(\operatorname{St}\tau - x) + x, \quad t \in (0,1).$$

Note that $V_{t,x}$ only intersects simplices in P that have τ as a face. In particular, $\tau_{t,x} \subset V_{t,x}$. If $\rho \in P$ and $\rho \subset \overline{\mathrm{St}}_x$, let

$$\rho_{t,x} := t(\rho - x) + x, \quad t \in (0,1). \tag{A.76}$$

Then, $\rho_{t,x} \subset \overline{\mathrm{St}}_x$ is a simplex, for $t \in (0,1)$

$$\{\rho_{t,x} : \rho \in P \text{ and } \rho \subset \overline{\mathrm{St}}_x\}$$

is a simplicial complex, call it $P_{t,x}$, and

$$\bar{V}_{t,x} = \bigcup_{\rho \in P;\, \rho \subset \overline{\mathrm{St}}_x} \rho_{t,x}.$$

Thus, $\bar{V}_{t,x}$ is the underlying space of $P_{t,x}$.

Since $\mathcal{C} \cap (\mathrm{Bd}\,\sigma)$ is closed by (2.6) and $x \notin \mathcal{C} \cap (\mathrm{Bd}\,\sigma)$, we may pick $t_x \in (0,1)$ so small that

$$\bar{V}_{t_x,x} \cap \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big] = \varnothing. \tag{A.77}$$

By (B.8), we have that $V_{t_x,x}$ is an open neighborhood of x. Hence, it suffices to prove that g is Lipschitz on $V_{t_x,x}$ for every $x \in |P| \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big]$.

Let $y_1, y_2 \in V_{t_x,x}$. Then, by definition of $\mathrm{St}\,\tau$ (appendix B), $y_i \in \mathrm{Int}\,\rho_{i;t_x,x}$ for some $\rho_{i;t_x,x} \in P_{t_x,x}$, s.t. $\rho_i \in P$ has τ as a face ($i = 1,2$). Thus, $\varnothing \neq \tau_{t_x,x} \subset \rho_{1;t_x,x} \cap \rho_{2;t_x,x}$. Suppose neither $\rho_{1;t_x,x}$ nor $\rho_{2;t_x,x}$ is a subset of the other. Then, by corollary B.3, there exists $K_x < \infty$, depending only on $\rho_{i;t_x,x}$ ($i = 1,2$), and $\tilde{y}_1, \tilde{y}_2 \in \mathrm{Int}\,(\rho_{1;t_x,x} \cap \rho_{2;t_x,x})$ s.t.

$$|y_2 - \tilde{y}_2| + |\tilde{y}_2 - \tilde{y}_1| + |\tilde{y}_1 - y_1| \leq K_x |y_2 - y_1|. \tag{A.78}$$

Now,

$$\big|g(y_2) - g(y_1)\big| \leq \big|g(y_2) - g(\tilde{y}_2)\big| + \big|g(\tilde{y}_2) - g(\tilde{y}_1)\big| + \big|g(\tilde{y}_1) - g(y_1)\big|.$$

Therefore, by (A.78), it suffices to prove that g is Lipschitz on $\rho_{t_x,x}$ for each $\rho \in P$ having τ as a face (which means in particular $x \in \rho$).

First, let $\rho \in P$ have τ as a face but assume $\rho \cap (\mathrm{St}\,\sigma) = \varnothing$. Let $y \in \rho_{t_x,x} \subset \rho \setminus \big[\mathcal{C} \cap (\mathrm{Bd}\,\sigma)\big]$. If $y \notin \overline{\mathrm{St}}\,\sigma$ or $y \in \mathrm{Lk}\,\sigma$, then, by (A.25), $g(y) = y$. Suppose $y \in (\overline{\mathrm{St}}\,\sigma) \setminus (\mathrm{Lk}\,\sigma)$ and let ξ be the face of ρ s.t. $y \in \mathrm{Int}\,\xi$. Since $\rho \cap (\mathrm{St}\,\sigma) = \varnothing$, we have that σ is not a face of ξ. Since $y \in \overline{\mathrm{St}}\,\sigma$, by (B.6), we have $\xi \subset \overline{\mathrm{St}}\,\sigma$. Therefore, $\xi \subset \overline{\mathrm{St}}\,\sigma$, $\xi \cap \sigma \neq \varnothing$, but σ is not a face of ξ. Hence, by (A.18) and (A.25) again, $g(y) = y$, since $y \notin \mathrm{Lk}\,\sigma$. In summary, $\rho \cap (\mathrm{St}\,\sigma) = \varnothing$ implies $g(y) = y$ for $y \in \rho_{t_x,x}$ so g is Lipschitz on $\rho_{t_x,x}$.

Next, let $\rho \in P$ have τ as a face (so $x \in \rho_{t_x,x} \subset \rho$) but this time assume $\rho \cap (\mathrm{St}\,\sigma) \neq \varnothing$. It follows from (B.6) and the definition of $\mathrm{St}\,\sigma$ (appendix B) that ρ has σ as a face. Let $\omega \in P$ be the face of ρ opposite σ. Let

$$U^\sigma := U^{\sigma,\rho} := \big\{y \in \rho : \mu(y) < 3/4\big\}.$$

By (A.23), U^σ is open in ρ.

Define

(A.79) $\ell(y) :=$
$$\begin{cases} \big[1-\mu(y)\big] + \mu(y)\left[\Big(1 - b\big[\sigma(y)\big]\Big) + b\big[\sigma(y)\big]\frac{\bar{\Delta}\left[\sigma(y)\right]}{diam(\sigma)}\right], & \text{if } y \in \rho \setminus \big[(\operatorname{Lk}\sigma) \cup \{z\}\big], \\ 1, & \text{if } y \in \rho \cap (\operatorname{Lk}\sigma), \\ 1, & \text{if } y = z. \end{cases}$$

Thus, $\ell \geq 0$. *Claim:*

$$\ell \text{ is continuous on } \rho \text{ and } \ell(y) = 0 \text{ if and only if } y \in \mathcal{C} \cap (\operatorname{Bd}\sigma). \tag{A.80}$$

To see this, note that (A.23) implies μ is continuous on ρ and lemma A.2(part 7), (A.11), tells us that b is continuous on σ. Lemma A.2(part 7) also tells use that $\bar{h}(x)$ is locally Lipschitz in $x \in \sigma \setminus \{z\}$. Therefore, (2.30), example C.1, and (2.6) imply that $\bar{\Delta}$ is continuous on $\sigma \setminus \{z\}$. But $\bar{\Delta}/diam(\sigma) \leq 1$ is bounded (see (2.31)) and (A.15) and(2.10) tells us $b\big[\sigma(z)\big] = b(z) = 0$. (A.16) tells us that $\sigma(\cdot)$ is continuous on $\rho \setminus (\operatorname{Lk}\sigma)$. But $\sigma(\cdot)$ is bounded and (A.22) tells us that $\mu(y) = 0$ on $\operatorname{Lk}\sigma$ and $\mu(z) = 1$. That establishes the continuity of ℓ.

(A.22) tells us that $\mu(y) = 1$ if and only if $y \in \sigma$, (A.15) tells us that $\sigma(y) = y$ if $y \in \sigma$, (2.10) tells us that $b(x) = 1$ if and only if $x \in \operatorname{Bd}\sigma$, and (2.30) and (2.6) tells us that $x \in \operatorname{Bd}\sigma$ and $\bar{\Delta}(x) = 0$ if and only if $x \in \mathcal{C} \cap (\operatorname{Bd}\sigma)$. Therefore, $\ell(y) = 0$ if and only if $y \in \mathcal{C} \cap (\operatorname{Bd}\sigma)$. This proves the claim (A.80).

For $\epsilon \in (0,1)$, let

$$U_{\sigma,\epsilon} := U_{\sigma,\epsilon,\rho} := \big\{ y \in \rho : \mu(y) > 1/4 \text{ and } \ell(y) > \epsilon \big\}.$$

By (A.23), (A.22), and (A.80), $U_{\sigma,\epsilon}$ is open in ρ and $U_{\sigma,\epsilon}$ contains $z \in \sigma$. In addition, by (A.22), (A.23), and (A.80),

$$\bar{U}_{\sigma,\epsilon} \text{ is compact and disjoint from } \operatorname{Lk}\sigma \text{ and } \mathcal{C} \cap (\operatorname{Bd}\sigma). \tag{A.81}$$

Note that, since $x \in \tau \subset \rho$, we have by (A.77) and (A.22),

$$\rho_{t_x,x} \subset \rho \setminus \big[\mathcal{C} \cap (\operatorname{Bd}\sigma)\big] = U^\sigma \cup \left(\bigcup_{\epsilon > 0} U_{\sigma,\epsilon} \right) \subset (\overline{\operatorname{St}}\,\sigma) \setminus \big[\mathcal{C} \cap (\operatorname{Bd}\sigma)\big].$$

In particular, either $x \in U^{\sigma,\rho}$ or, for some $\epsilon > 0$, $x \in U_{\sigma,\epsilon,\rho}$. Therefore, by compactness of $\overline{\rho_{t_x,x}}$ and (A.77), for each of the finitely many $\rho \in P$ having τ as a face and s.t. $\sigma \subset \rho$, we have either $\rho_{t_x,x} \subset U^{\sigma,\rho}$ or, for some $\epsilon > 0$, we have $\rho_{t_x,x} \subset U_{\sigma,\epsilon,\rho}$. Thus, it suffices to show that g is Lipschitz on each $U^{\sigma,\rho}$ and on each $U_{\sigma,\epsilon,\rho}$.

Let $\rho \in P$ with $\tau \subset \rho \subset \overline{\operatorname{St}}_x$ and $\sigma \subset \rho$ be fixed. We show that g is Lipschitz in U^σ. The closure $\bar{U}^\sigma$ of $U^{\sigma,\rho}$ is compact and lies, by (A.22) and (A.23), in $(\overline{\operatorname{St}}\,\sigma) \setminus \sigma$. In addition, $\sigma \times [1/4, 1]$ is a compact subset of B_z as defined in (A.12). This means, by (A.23) and lemma A.2(part 9), that

$$\mu \text{ and } w \text{ are Lipschitz on } U^\sigma. \text{ Moreover, } s_z \text{ is Lipschitz on } \sigma \times [1/4, 1]. \tag{A.82}$$

Let $y_1, y_2 \in U^\sigma$. Recall that ω is the face of ρ opposite σ. So $\omega \subset U^\sigma$ by (A.22). To simplify the notation, write $x_i = \sigma(y_i)$, $\mu_i = \mu(y_i)$, $s_i = s(x_i, 1 - \mu_i)$, and $w_i = w(y_i) \in \omega$ $(i = 1, 2)$. (See (A.20).) Also, $K_1, K_2, \ldots$ will be finite and only depend on U^σ. Then

$$\big|g(y_2) - g(y_1)\big| \leq |\mu_2 s_2 - \mu_1 s_1| + \big|(1-\mu_2)w_2 - (1-\mu_1)w_1\big| \leq |\mu_2 s_2 - \mu_1 s_1| + K_1 |y_2 - y_1|,$$

by (A.24), (C.8), and (A.82). Let $D := \max\{|u|, u \in |P|\} < \infty\}$. Then, also by (A.82),

$$\begin{aligned}|\mu_2 s_2 - \mu_1 s_1| &\leq |\mu_2 s_2 - \mu_2 s_1| + |\mu_2 s_1 - \mu_1 s_1| \\ &= \mu_2 |s_2 - s_1| + |\mu_2 - \mu_1||s_1| \\ &\leq \mu_2 K_2 \big|(x_2, 1-\mu_2) - (x_1, 1-\mu_1)\big| + DK_3|y_2 - y_1| \\ &\leq \mu_2 K_2 \big(|x_2 - x_1| + |\mu_1 - \mu_2|\big) + DK_3|y_2 - y_1| \\ &\leq \mu_2 K_2 |x_2 - x_1| + K_2 K_4 |y_2 - y_1| + DK_3 |y_2 - y_1|.\end{aligned}$$

Thus, to prove that g is Lipschitz in U^σ it suffices to find a $K < \infty$ s.t.

$$\mu_2 |x_2 - x_1| \leq K|y_2 - y_1|. \tag{A.83}$$

Note that $\min\{dist(y, \omega), y \in \sigma\} > 0$. Then, from (A.19), we see

$$\mu_2 = \frac{|y_2 - w_2|}{|x_2 - w_2|} \leq \frac{|y_2 - w_2|}{\min\{dist(x, \omega), x \in \sigma\}}.$$

Thus, to prove (A.83), it suffices to find $K_5 < \infty$ s.t.

$$|y_2 - w_2||x_2 - x_1| \leq K|y_2 - y_1|. \tag{A.84}$$

Let

$$y_1' = \mu_1 x_1 + (1 - \mu_1) w_2 \in \rho.$$

So, by (A.19), $\sigma(y_1') = x_1$ (even if $y \in \operatorname{Lk}\sigma$; see (A.20)) and $\mu(y_1') = \mu_1$. In particular, $y_1' \in U^\sigma$ because $y_1 \in U^\sigma$. Now, by (A.82) we have that $w_i = w(y_i)$ is Lipschitz in $y_i \in U^\sigma$ $(i = 1, 2)$. Therefore, for some $K_6 < \infty$,

$$\begin{aligned}|y_2 - y_1'| &\leq |y_2 - y_1| + \big|y_1 - y_1'\big| \\ &= |y_2 - y_1| + (1 - \mu_1)|w_1 - w_2| \\ &= |y_2 - y_1| + K_6 |y_2 - y_1|.\end{aligned} \tag{A.85}$$

Thus, to prove (A.84) it suffices to show that there exists $K_7 < \infty$ s.t.

$$|y_2 - w_2||x_2 - x_1| \leq K_7 |y_2 - y_1'|. \tag{A.86}$$

But we can "trick" lemma A.2(part 6) into doing this for us. The point w_2 will play the role of z in lemma A.2(part 6). But ρ cannot play the role of σ in lemma A.2(part 6) because $z \in \operatorname{Int}\sigma$ (by (2.4)) while $w_2 \in \operatorname{Bd}\rho$. So we have to replace ρ by a bigger simplex that contains w_2 in its interior. This is accomplished by the following lemma.

Lemma A.5. *Let χ be a simplex, let ξ be a face of χ, and let ζ be the face of χ opposite ξ. Then one can construct from χ and ξ a simplex χ' s.t. ξ is a face of χ' and $\zeta \subset Int\chi'$. In particular, $\chi \subset \chi'$.*

Proof. Let $\hat{\chi}$ be the barycenter of χ ((B.3)) and modify the vertices of χ as follows.

$$v' = \begin{cases} v, & \text{if } v \in \xi^{(0)}, \\ 2(v - \hat{\chi}) + \hat{\chi} = 2v - \hat{\chi}, & \text{if } v \in \zeta^{(0)}. \end{cases} \tag{A.87}$$

We show that $v'(v \in \chi^{(0)})$ are geometrically independent (appendix B). Let $c_v \in \mathbb{R}$ $(v \in \chi^{(0)})$ satisfy

$$\sum_{v\in\chi^{(0)}} c_v = 0 \text{ and } \sum_{v\in\chi^{(0)}} c_v v' = 0. \tag{A.88}$$

Let m be the number of vertices in χ and let $r < m$ be the number of vertices in ζ. Then, by (A.87) and (A.88)

$$\begin{aligned}
0 &= \sum_{\xi^{(0)}} c_v v + \sum_{\zeta^{(0)}} 2c_v v - \sum_{v\in\chi^{(0)}} \left(\frac{1}{m} \sum_{u\in\zeta^{(0)}} c_u \right) v \\
&= \sum_{\xi^{(0)}} \left(c_v - \frac{1}{m} \sum_{\zeta^{(0)}} c_u \right) v + \sum_{\zeta^{(0)}} \left(2c_v - \frac{1}{m} \sum_{\zeta^{(0)}} c_u \right) v \\
&= \sum_{\xi^{(0)}} \left(c_v + \frac{1}{m} \sum_{\xi^{(0)}} c_u \right) v + \sum_{\zeta^{(0)}} \left(2c_v - \frac{1}{m} \sum_{\zeta^{(0)}} c_u \right) v.
\end{aligned}$$

Also by (A.88), when we add up the coefficients in the preceding we get,

$$\begin{aligned}
\sum_{\xi^{(0)}} &\left(c_v + \frac{1}{m} \sum_{\xi^{(0)}} c_u \right) + \sum_{\zeta^{(0)}} \left(2c_v - \frac{1}{m} \sum_{\zeta^{(0)}} c_u \right) \\
&= \sum_{\xi^{(0)}} c_v + \frac{m-r}{m} \sum_{\xi^{(0)}} c_v + 2\sum_{\zeta^{(0)}} c_v - \frac{r}{m} \sum_{\zeta^{(0)}} c_v \\
&= \frac{m-r}{m} \sum_{\xi^{(0)}} c_v + \left(1 - \frac{r}{m}\right) \sum_{\zeta^{(0)}} c_v \\
&= 0.
\end{aligned}$$

Therefore, since the vertices, $v \in \chi^{(0)}$, are geometrically independent, we have, by (A.88),

$$c_v = -\frac{1}{m} \sum_{\xi^{(0)}} c_u = \frac{1}{m} \sum_{\zeta^{(0)}} c_u, \ (v \in \xi^{(0)}) \text{ and } c_v = \frac{1}{2m} \sum_{\zeta^{(0)}} c_u, \ (v \in \zeta^{(0)}). \tag{A.89}$$

But $\sum_{v\in\chi^{(0)}} c_v = 0$, by (A.88). Hence, (A.89) implies that a non-zero multiple of $\sum_{u\in\zeta^{(0)}} c_u$ is 0. Therefore, by (A.89) again, $c_v \propto \sum_{u\in\zeta^{(0)}} c_u = 0$ for every $v \in \chi^{(0)}$. This proves the geometric independence of v' $(v \in \chi^{(0)})$.

Let χ' be the simplex with vertex set $\{v', v \in \chi^{(0)}\}$. Then by (A.87),

$$v = \tfrac{1}{2} v' + \tfrac{1}{2} \hat{\chi}, \quad \text{if } v \in \zeta^{(0)}. \tag{A.90}$$

It easily follows that

$$\hat{\chi} = \frac{2}{2m-r} \sum_{v\in\xi^{(0)}} v' + \frac{1}{2m-r} \sum_{v\in\zeta^{(0)}} v'.$$

In particular, $\hat{\chi}$ is an interior point of χ'. Hence, by (A.90), each $v \in \zeta^{(0)}$ is an interior point of χ'. Hence, $\zeta \subset \operatorname{Int} \chi'$. Since $\hat{\chi} \in \operatorname{Int} \chi$, (A.87) and (A.90) imply that $v \in \chi'$ for every $v \in \chi^{(0)}$. Finally, every vertex of ξ is a vertex of χ' by (A.87). Therefore, ξ is a face of χ'. □

Proof of lemma A.4, continued: Apply lemma A.5 with $\chi = \rho$ and $\xi = \sigma$, so $\zeta = \omega$. (Recall that ω is the face of ρ opposite σ.) Thus, there is a simplex ρ' s.t. $\omega \subset \operatorname{Int} \rho'$, $\rho \subset \rho'$, and σ is a face of ρ'. In particular, $w_2 \in \operatorname{Int} \rho'$ and $y_1', y_2 \in \rho'$. By (2.8), (A.19), and lemma A.2(part 6),

$$|y_2 - w_2||x_2 - x_1| = |y_2 - w_2|\big|\bar{h}_{w_2,\rho'}(y_2) - \bar{h}_{w_2,\rho'}(y_1')\big| \leq K''(w_2)|y_2 - y_1'|.$$

Now, $K''(w_2)$ is continuous in w_2, by lemma A.2(part 6). Therefore, by (A.82), $K''(w_2) = K''\big[w(y_2)\big]$ is bounded in $y_2 \in U^\sigma$. I.e., (A.86) holds.

Next, let $\epsilon \in (0,1)$ and consider $U_{\sigma,\epsilon} = U_{\sigma,\epsilon,\rho}$. By (A.81), (A.16), and(A.23), and we have

$$\sigma(\cdot) \text{ and } \mu \text{ are Lipschitz on } \bar{U}_{\sigma,\epsilon}. \tag{A.91}$$

Moreover, the map $\sigma(\cdot) \times (1-\mu)$ maps $\bar{U}_{\sigma,\epsilon}$ onto a compact subset of B_z. (See (A.12), (A.15), and (A.22).) Hence, by lemma A.2(part 9),

$$s \text{ is Lipschitz on } \big[\sigma(\cdot) \times (1-\mu)\big](\bar{U}_{\sigma,\epsilon}). \tag{A.92}$$

Let $y_1, y_2 \in U_{\sigma,\epsilon}$. Write $x_i = \sigma(y_i)$, $\mu_i = \mu(y_i)$, $s_i = s(x_i, 1-\mu_i)$, and $w_i = w(y_i)$ $(i = 1,2)$. WLOG $\mu_2 \geq \mu_1$

Recall $D := \max\{|u|, u \in |P|\} < \infty\}$. Then we have, by (A.24), (A.92), and (A.91),

$$\begin{aligned}
\big|g(y_2) - g(y_1)\big| &\leq |\mu_2 s_2 - \mu_1 s_1| + \big|(1-\mu_2)w_2 - (1-\mu_1)w_1\big| \\
&\leq \big|\mu_2(s_2 - s_1)\big| + |\mu_2 s_1 - \mu_1 s_1| + (1-\mu_2)|w_2 - w_1| + |\mu_1 - \mu_2||w_1| \\
&= \mu_2|s_2 - s_1| + |\mu_2 - \mu_1||s_1| + |\mu_1 - \mu_2||w_1| + (1-\mu_2)|w_2 - w_1| \\
&\leq K_1\big(|x_2 - x_1| + |\mu_1 - \mu_2|\big) + K_2 D|y_2 - y_1| \\
&\qquad + K_2 D|y_2 - y_1| + (1-\mu_2)|w_2 - w_1|.
\end{aligned} \tag{A.93}$$

By (A.91) again,

$$K_1\big(|x_2 - x_1| + |\mu_1 - \mu_2|\big) \leq K_1\big(K_3|y_2 - y_1| + K_2|y_2 - y_1|\big).$$

So, by (A.93), to prove that g is Lipschitz on $\bar{U}_{\sigma,\epsilon}$ it suffices to find $K < \infty$ s.t.

$$(1-\mu_2)|w_2 - w_1| \leq K|y_2 - y_1|. \tag{A.94}$$

By (A.19),

$$(1-\mu_2)(w_2 - x_2) = y_2 - x_2.$$

Recall that ω is the face of ρ opposite σ. I.e.

$$1 - \mu_2 = \frac{|y_2 - x_2|}{|w_2 - x_2|} \leq \frac{|y_2 - x_2|}{\min\{|w - x| : w \in \omega, x \in \sigma\}}. \tag{A.95}$$

Let

$$y_1' = \mu_1 x_2 + (1-\mu_1)w_1 \in \rho.$$

Let $\Delta_i = \bar{\Delta}(x_i)$ $(i = 1,2)$. Now, by (A.19), $\sigma(y_1') = x_2$ and $\mu(y_1') = \mu_1 > 1/4$. Hence, $\bar{\Delta}\big[\sigma(y_1')\big] = \Delta_2$ and $b\big[\sigma(y_1')\big] = b\big[\sigma(y_2)\big]$.

Now, by (A.22), $y_2 \in U_{\sigma,\epsilon}$ implies $y_2 \notin \operatorname{Lk} \sigma$. Suppose $y_2 = z \in \sigma$. Then, by (A.15), $\sigma(y_1') = \sigma(y_2) = z$. Therefore, by (2.10), $b\big[\sigma(y_1')\big] = 0$ so, by (A.79) (whether $y_1' = z$, $y_1' \in \operatorname{Lk} \sigma$, or neither), we have $\ell(y_1') = 1$ so $y_1' \in U_{\sigma,\epsilon}$ if $y_2 = z$.

Suppose $y_2 \neq z$. Then we have, by definition of $U_{\sigma,\epsilon}$ and the facts that $y_2 \notin \operatorname{Lk} \sigma$ and $\mu_2 \geq \mu_1$,

$$\begin{aligned}
\epsilon < \ell(y_2) &= (1-\mu_2) + \mu_2 \left[\Big(1 - b(x_2)\Big) + b(x_2) \frac{\Delta_2}{diam(\sigma)} \right] \\
&= 1 - \mu_2\, b(x_2) \left(1 - \frac{\Delta_2}{diam(\sigma)}\right) \\
&\leq 1 - \mu_1\, b(x_2) \left(1 - \frac{\Delta_2}{diam(\sigma)}\right) \\
&= \ell(y_1'),
\end{aligned}$$

since $\Delta_2/diam(\sigma) \leq 1$ and, by (A.22), $\mu_1 = \mu(y_1') = 0$ if $y_1 \in \operatorname{Lk}\sigma$. (See (2.30) and (2.31).) Therefore, again $y_1' \in U_{\sigma,\epsilon}$. Now by (A.19) and (A.91),

$$\begin{aligned}
|y_2 - y_1'| &\leq |y_2 - y_1| + |y_1 - y_1'| \\
&= |y_2 - y_1| + \mu_1 |x_1 - x_2| \\
&\leq (1 + K_3)|y_2 - y_1|.
\end{aligned} \tag{A.96}$$

Hence, by (A.95), to prove (A.94) it suffices to find $K_4 < \infty$ s.t.

$$|y_2 - x_2||w_2 - w_1| \leq K_4 |y_2 - y_1'|. \tag{A.97}$$

To prove (A.97), apply lemma A.5 with $\chi = \rho$ and $\xi = \omega$, so $\zeta = \sigma$ to enlarge ρ to a simplex ρ' (not in P) s.t. $\sigma \subset \operatorname{Int}\rho'$ while ω remains a face of ρ'. Then apply (A.19), (2.8) (identifying $1-\mu$ and b), and lemma A.2(part 6) as follows.

$$|y_2 - x_2||w_2 - w_1| = |y_2 - x_2| \big| \bar{h}_{x_2,\rho'}(y_2) - \bar{h}_{x_2,\rho'}(y_1') \big| \leq K''(x_2)|y_2 - y_1'|.$$

Now, $K''(x_2)$ is continuous in x_2, by lemma A.2(part 6). Therefore, by (A.91), $K''(x_2) = K''\big[\sigma(y_2)\big]$ is bounded in $y_2 \in U_{\sigma,\epsilon}$. I.e., (A.97) holds. This completes the proof that g is Lipschitz on $U_{\sigma,\epsilon}$. Lemma A.4 is proved. □

The following was adapted from [Ella, Appendix A].

Lemma A.6. *If M is a symmetric $q \times q$ (real) matrix (q, a given positive integer), let $\Lambda(M) = (\lambda_1(M), \ldots, \lambda_q(M))$, where $\lambda_1(M) \geq \ldots \geq \lambda_q(M)$ are the eigenvalues of M. Let also $\|M\|$ be the Frobenius norm (Blum* et al *[BCSS98, p. 203]) $\|M\| = \sqrt{trace\, MM^T}$. Then Λ is a continuous function (w.r.t. $\|\cdot\|$). Moreover, if N and $M_1, M_2, \ldots$ are all symmetric $q \times q$ (real) matrices s.t. $M_j \to N$ (w.r.t. $\|\cdot\|$, i.e., entrywise) as $j \to \infty$, let $Q_j^{q\times q}$ be a matrix whose rows comprise an orthonormal basis of $\mathbb{R}^q$ consisting of eigenvectors of M_j. Then there is a subsequence $j(n)$ s.t. $Q_{j(n)}$ converges to a matrix whose rows comprise a basis of $\mathbb{R}^q$ consisting of unit eigenvectors of N.*

Proof. Let N and $M_1, M_2, \ldots$ all be symmetric $q \times q$ matrices. Suppose $M_j \to N$ as $j \to \infty$. Let $\mu_i = \lambda_i(N)$ ($i = 1, \ldots, q$). Since $M_j \to N$, $\{M_j\}$ is bounded. Hence, $\big\{(\lambda_1(M_j), \ldots, \lambda_q(M_j))\big\}$ is bounded (Marcus and Minc [MM64, 1.3.1, pp. 140–141]). For each n, let $v_{n1}, \ldots, v_{nq} \in \mathbb{R}^q$ be orthonormal eigenvectors of M_n corresponding to $\lambda_1(M_n), \ldots, \lambda_q(M_n)$, resp. In particular, $v_{n1}, \ldots, v_{nq}$ span $\mathbb{R}^q$. Let S^{q-1} be the $(q-1)$-sphere

$$S^{q-1} = \{x \in \mathbb{R}^q : |x| = 1\}.$$

Then $v_{n1}, \ldots, v_{nq} \in S^{q-1}$. By compactness of S^{q-1} we may choose a subsequence, $\{j(n)\}$, s.t. $v_{j(n)i}$ converges to some $v_i \in \mathbb{R}^q$ and $\lambda_i(M_{j(n)})$ converges to some $\nu_i \in \mathbb{R}$ as $n \to \infty$ ($i = 1, \ldots, q$). Then $v_1, \ldots, v_q$ are orthonormal eigenvectors of N with eigenvalues $\nu_1 \geq \ldots \geq \nu_q$. Hence, $\{\nu_1, \ldots, \nu_q\} \subset \{\mu_1, \ldots, \mu_q\}$. Note that $v_1, \ldots, v_q$ must span $\mathbb{R}^q$.

We show that

$$\nu_i = \mu_i \quad (i = 1, \ldots, q). \tag{A.98}$$

This is obvious if $q = 1$. Suppose (A.98) always holds for $q = m \geq 1$ and suppose $q = m + 1$. Suppose some $\mu_i \notin \{\nu_1, \ldots, \nu_q\}$ and let $w_i \in \mathbb{R}^q \setminus \{0\}$ be an eigenvector of N corresponding to μ_i. Then by Stoll and Wong [SW68, Theorem 4.1, p. 207], we have that $w_i \perp v_1, \ldots, v_q$. But $v_1, \ldots, v_q$ span $\mathbb{R}^q$. Hence, $w_i = 0$, contradiction. Thus, for some j_i we have $\mu_i = \nu_{j_i}$.

Let $T : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ be the linear operator associated with N. Let $V \subset \mathbb{R}^q$ be the orthogonal complement of the span of $v_{j_i} = w_i$ and let N_i be a matrix of the restriction $T|_V$. Then V is spanned by v_j with $j \neq j_i$ and $\mu_{i'}$ with $i' \neq i$ are the eigenvalues of N_i. Then by the induction hypothesis $j_i = i$ and $\nu_{i'} = \mu_{i'}$ for $i \neq i'$, etc. Thus, $\{\mu_1, \ldots, \mu_q\} = \{\nu_1, \ldots, \nu_q\}$.

The preceding argument obviously goes through it we had started with a subsequence of $M_1, M_2, \ldots$. Thus, any subsequence has a further subsequence s.t. $\Lambda(M_n)$ converges to $\Lambda(N)$ along that subsequence of a subsequence. It follows that $\Lambda(M_n) \to \Lambda(N)$. I.e., Λ is continuous. □

B. Basics of simplicial complexes

This appendix presents some of the material in Munkres [Mun84], mostly from pages 2 – 11, 83, and 371. (See also Rourke and Sanderson [RS72].) Let N be a positive integer and let $n \in \{0, \ldots, N\}$. Points $v(0), \ldots, v(n)$ in $\mathbb{R}^N$ are "geometrically independent" (or are in "general position") if $v(1) - v(0), \ldots, v(n) - v(0)$ are linearly independent. Equivalently, $v(0), \ldots, v(n)$ are geometrically independent if and only if

$$\sum_{i=1}^{n} t_i = 0 \text{ and } \sum_{i=1}^{n} t_i v(i) = 0$$

together imply $t_0 = \cdots t_n = 0$. If $v(0), \ldots, v(n) \in \mathbb{R}^N$ are geometrically independent then they are the vertices of the "simplex"

$$\sigma = \{\beta_0 v(0) + \cdots + \beta_n v(n) : \beta_0, \ldots \beta_n \geq 0 \text{ and } \beta_0 + \cdots + \beta_n = 1\}.$$

We say that σ is "spanned" by $v(0), \ldots, v(n)$ and n is the "dimension" of σ. (Sometimes we call σ a "n-simplex" and write $\dim \sigma = n$.) Note that σ is convex and compact. Indeed, it is the convex hull of $\{v(0), \ldots, v(n)\}$. Thus, every $y \in \sigma$ can be expressed uniquely (and continuously) in "barycentric coordinates"

$$y = \sum_{v \text{ is a vertex in } \sigma} \beta_v(y)\, v,$$

where the $\beta_v(y)$'s are nonnegative and sum to 1.

The simplex σ lies on the plane

$$\Pi = \{\beta_0 v(0) + \cdots + \beta_n v(n) : \beta_0 + \cdots + \beta_n = 1\}. \tag{B.1}$$

I.e., the definition of Π is like that of σ except the non-negativity requirement is dropped. Note that Π need not include the origin of $\mathbb{R}^N$. Π is the smallest plane containing σ. The dimension

of Π is n. Any subset of $\{v(0), \ldots, v(n)\}$ is geometrically independent and the simplex spanned by that subset is a "face" of σ. So σ is a face of itself and a vertex of σ is also a face of σ (and so is ?). A "proper" face of σ is a face of σ different from σ.

Let σ be a simplex spanned by geometrically independent points $v(0), \ldots, v(n)$. If $J \subsetneqq \{0, \ldots, n\}$ is nonempty, let τ be the proper face of σ spanned by $\{v(j),\, j \in J\}$. The face "opposite" τ is the span, ω, of $\{v(j),\, j \notin J\}$ (Munkres [Mun84, p. 5 and Exercise 4, p. 7]). E.g., τ might be a vertex of σ. Thus, τ consists of those $y \in \sigma$ s.t. $\beta_v(y) = 0$ for all vertices $v \notin \tau$ and ω consists of those $y \in \sigma$ s.t. $\beta_v(y) = 0$ for all vertices $v \in \tau$.

The union of all proper faces of σ is the "boundary" of σ, denoted $\operatorname{Bd} \sigma$. The "(simplicial) interior" of σ (as a simplex) is the set $\operatorname{Int} \sigma := \sigma \setminus (\operatorname{Bd} \sigma)$, where "\" indicates set-theoretic subtraction. If

$$y = \sum_{j=0}^{n} \beta_j(y)\, v(j), \text{ then } y \in \operatorname{Int} \sigma \text{ if and only if } \beta_j(y) > 0 \text{ for all } j = 0, \ldots, n. \tag{B.2}$$

Thus, the interior of σ as a simplex is in general different from its (usually empty) interior as a subspace of $\mathbb{R}^N$. In fact, the interior (as a simplex) of a 0-dimensional simplex (a single point) is the point itself. But σ is the topological closure of $\operatorname{Int} \sigma$ and $\operatorname{Int} \sigma$ is the relative interior of σ as a subset of Π defined by (B.1).

The following lemma (Munkres [Mun84, lemma 1.1, p. 6]) about convex sets is handy.

Lemma B.1. *Let U be a bounded, convex, open set in some affine space (e.g., a Euclidean space). Let $w \in U$. Then each ray emanating from w intersects the boundary of U in precisely one point.*

Let $v(0), \ldots, v(n) \in \mathbb{R}^n$ be the vertices of σ. Then

$$\hat{\sigma} = \frac{1}{n+1} \sum_{j=0}^{n} v(j) \tag{B.3}$$

is the "barycenter" of σ (Munkres [Mun84, p. 85]). Munkres [Mun66, p. 90] defines the "radius", $r(\sigma)$, of σ to be the minimum distance from $\hat{\sigma}$ to $\operatorname{Bd} \sigma$. He defines the "thickness" of the simplex σ to be $t(\sigma) := r(\sigma)/diam(\sigma)$. Here, "$diam(\sigma)$" is the diameter of σ, i.e., the length of the longest edge of σ.

A "simplicial complex", P, in $\mathbb{R}^N$ is a collection of simplices in $\mathbb{R}^N$ s.t.

$$\text{Every face of a simplex in } P \text{ is in } P. \tag{B.4}$$

and

$$\text{The intersection of any two simplices in } P \text{ is a face of each of them.} \tag{B.5}$$

It turns out that an equivalent definition of simplicial complex is obtained by replacing condition (B.5) by the following.

$$\text{Every pair of distinct simplices in } P \text{ have disjoint interiors.} \tag{B.5'}$$

It follows that

$$\text{If } \rho, \sigma \text{ are elements of a simplicial complex and } (\operatorname{Int} \rho) \cap \sigma \neq \varnothing \text{ then } \rho \text{ is a face of } \sigma. \tag{B.6}$$

(*Proof:* $(\operatorname{Int} \rho) \cap \sigma$ lies in some face of σ, e.g., in σ itself. Let τ be the smallest face of σ (in terms of inclusion) containing $(\operatorname{Int} \rho) \cap \sigma$. Suppose $\rho \neq \tau$. Then by (B.5') $(\operatorname{Int} \rho) \cap \tau$ lies in

some proper face of τ. But $(\operatorname{Int}\rho)\cap\tau = \big[(\operatorname{Int}\rho)\cap\sigma\big]\cap\tau = (\operatorname{Int}\rho)\cap\sigma$, since $(\operatorname{Int}\rho)\cap\sigma \subset \tau \subset \sigma$. I.e., $(\operatorname{Int}\rho)\cap\sigma$ lies in a proper face of τ. That contradicts the minimality of τ. Therefore, $\rho = \tau$.)

A simplicial complex, P—, is "finite" if it is finite as a set (of simplices). The "dimension" of a simplicial complex is

$$\dim P = \max\{\dim\sigma : \sigma \in P\}$$

(Munkres [Mun84, p. 14]). (So infinite dimensional simplicial complexes are possible.) In the following assume P is a non-empty simplicial complex.

A subset, L, of P is a "subcomplex" of P if L is a simplicial complex in its own right. The collection, $P^{(q)}$, of all simplices in P of dimension at most $q \geq 0$ is a subcomplex, called the "q-skeleton" of P. In particular, $P^{(0)}$ is the set of all vertices of simplices in P. The "polytope" or "underlying space" of P, denoted by $|P|$, is just the union of the simplices in P. If P is finite, i.e., consists of finitely many simplices, then $|P|$ is assigned the relative topology it inherits from $\mathbb{R}^N$. In general, a subset, X, of $|P|$ is closed (open) if and only if $X \cap \sigma$ is closed (resp. open) in σ for every $\sigma \in P$. Call this topology the "polytope topology" on $|P|$. A space that equals $|P|$ for some simplicial complex, P, is called a "polyhedron". By Munkres [Mun84, Lemma 2.5, p. 10], $|P|$ is compact if and only if P is finite. If X is a topological space, then a "triangulation" of X is a simplicial complex, P, and a homeomorphism $f : |P| \to X$.

Let P be a finite simplicial complex of positive dimension. As in Munkres [Mun84, p. 10], define "barycentric coordinates" on $|P|$ as follows. First, note that

(B.7) If $x \in |P|$ then there is exactly one simplex $\tau \in P$ s.t. $x \in \operatorname{Int}\tau$.

(To see this, note that since P is finite, there is a smallest simplex (w.r.t. inclusion order), τ, in P containing x. Clearly, $x \in \operatorname{Int}\tau$. By (B.5') this implies τ is unique.) Let $\tau^{(0)}$ be the set of vertices of τ. Then, by (B.2), there exist strictly positive numbers $\beta_v(x)$, $v \in \tau^{(0)}$ that sum to 1 and satisfy

$$x = \sum_{v\in\tau^{(0)}} \beta_v(x)v.$$

Since $v \in \tau^{(0)}$ are geometrically independent, $\beta_v(x)$, $v \in \tau^{(0)}$, are unique. If $v \in P^{(0)}$ is not a vertex of τ define $\beta_v(x) = 0$. Thus,

$$x = \sum_{v\in P^{(0)}} \beta_v(x)v, \qquad x \in |P|.$$

The entries in $\{\beta_v(x),\ v \in P^{(0)}\}$ are the "barycentric coordinates" of x. For each $v \in P^{(0)}$ the function β_v is continuous on $|P|$ (Munkres [Mun84, p. 10]). If P is finite, we have the following. (See below for the proof.)

Lemma B.2. *Let P be a finite simplicial complex. Then the vector-valued function $\boldsymbol{\beta} : x \mapsto \{\beta_v(x),\ v \in P^{(0)}\}$ is Lipschitz in $x \in |P|$ (w.r.t. the obvious Euclidean metrics; see appendix C).*

In the course of proving this lemma, the following useful fact emerges.

Corollary B.3. *Let P be a finite simplicial complex. There exists $K < \infty$ s.t. the following holds. Let $\rho, \tau \in P$ satisfy $\rho \cap \tau \neq \varnothing$, but suppose neither simplex is a subset of the other. If $x \in \operatorname{Int}\rho$ and $y \in \operatorname{Int}\tau$ then there exist $\tilde{x}, \tilde{y} \in \operatorname{Int}(\rho\cap\tau)$ s.t.*

$$|x-\tilde{x}| + |\tilde{x}-\tilde{y}| + |\tilde{y}-y| \leq K|x-y|.$$

Let $\sigma \in P$ and let

$$\overline{\mathrm{St}}\,\sigma = \bigcup_{\sigma \subset \omega \in P} \omega.$$

$\overline{\mathrm{St}}\,\sigma$ is the "closed star" of σ (Munkres [Mun84, p. 371]). By (B.5) $\overline{\mathrm{St}}\,\sigma$ is the union of all simplices in P having σ as a face. In particular, $\sigma \subset \overline{\mathrm{St}}\,\sigma$. Let $\mathrm{Lk}\,\sigma$ be the union of all simplices in $\overline{\mathrm{St}}\,\sigma$ that do *not* intersect σ. $\mathrm{Lk}\,\sigma$ is the "link" of σ. The simplices in $\mathrm{Lk}\,\sigma$ will be faces of simplices in $\overline{\mathrm{St}}\,\sigma$ that also have σ as a face. We may have $\overline{\mathrm{St}}\,\sigma = \sigma$, which implies $\mathrm{Lk}\,\sigma = \varnothing$. This can happen, e.g., if $\dim \sigma = \dim P$. If $\rho \in P$, σ is a proper face of ρ, and ω is the face of ρ opposite σ, then $\omega \subset \mathrm{Lk}\,\sigma$. Thus, $\overline{\mathrm{St}}\,\sigma = \sigma$ if and only if $\mathrm{Lk}\,\sigma = \varnothing$.

Let the "star", $\mathrm{St}\,\sigma$ of σ be the union of the interiors of all simplices of P having σ as a face (Munkres [Mun84, p. 371]). (If $\overline{\mathrm{St}}\,\sigma = \sigma$, then $\mathrm{St}\,\sigma = \mathrm{Int}\,\sigma$.) We have

(B.8) $\mathrm{St}\,\sigma = \big\{ y \in |P| : \beta_v(y) > 0 \text{ for every } v \in \sigma^{(0)} \big\}$ so $\mathrm{St}\,\sigma$ is open in $|P|$.
Moreover, $\mathrm{Int}\,\sigma \subset \mathrm{St}\,\sigma, (\mathrm{St}\,\sigma) \cap (\mathrm{Lk}\,\sigma) = \varnothing,$ and $(\mathrm{St}\,\sigma) \cap (\mathrm{Bd}\,\sigma) = \varnothing.$

(*Proof:* $\rho \in P$ has σ as a face if and only if $\sigma^{(0)} \subset \rho^{(0)}$. But $x = \sum_{v \in \rho^{(0)}} \beta_v(x) \in \mathrm{Int}\,\rho$ if and only if $\beta_v(x) > 0$ for every $v \in \rho^{(0)}$. Hence, if $x \in \mathrm{Int}\,\rho$ and ρ has σ as a face then $\beta_v(x) > 0$ for every $v \in \sigma^{(0)}$. Conversely, suppose $x \in |P|$ and $\beta_v(x) > 0$ for every $v \in \sigma^{(0)}$. Then obviously, if ρ is the simplex in P with $x \in \mathrm{Int}\,\rho$, we have $\sigma^{(0)} \subset \rho^{(0)}$ so $\rho \in P$ has σ as a face. Thus, $x \in \mathrm{Int}\,\rho \subset \mathrm{St}\,\sigma$. In particular, $\mathrm{Int}\,\sigma \subset \mathrm{St}\,\sigma$. Since β_v ($v \in \sigma^{(0)}$) are continuous, continuous, if follows that $\mathrm{St}\,\sigma$ is open. Moreover, if $x \in (\mathrm{Lk}\,\sigma) \cup (\mathrm{Bd}\,\sigma)$ then $\beta_v(x) = 0$ for some $v \in \sigma^{(0)}$. Hence, neither $\mathrm{Lk}\,\sigma$ nor $\mathrm{Bd}\,\sigma$ intersects $\mathrm{St}\,\sigma$.)

Let $\sigma \in P$. Observe that, true to their names, both $\overline{\mathrm{St}}\,\sigma$ and $\mathrm{St}\,\sigma$ are "starlike" w.r.t. any $x \in \mathrm{Int}\,\sigma$. I.e., if $y \in \overline{\mathrm{St}}\,\sigma$ then the line segment joining x and y lies entirely in $\overline{\mathrm{St}}\,\sigma$. The same goes for $y \in \mathrm{St}\,\sigma$. *Claim:* $|P|$ is locally arcwise connected (Massey [Mas67, p. 56]). To see this, let $x \in |P|$ and let σ be the unique simplex in P s.t. $x \in \mathrm{Int}\,\sigma$. (See (B.7).) $\mathrm{St}\,\sigma$ is an open neighborhood of x. Let $r > 0$ be so small that the open ball $B_r(x)$, of radius r centered at x satisfies $B_r(x) \cap |P| \subset \mathrm{St}\,\sigma$. If $y, z \in B_r(x) \cap |P|$, then the line segments joining y to x and x to z also lie in $B_r(x) \cap |P|$. I.e., $B_r(x) \cap |P|$ is path connected. This proves the claim. Thus, if $|P|$ is connected it is also arcwise connected.

The following result will be helpful in proving corollary 1.2.

Lemma B.4. *Let P be a simplicial complex lying in a finite dimensional Euclidean space, $\mathbb{R}^N$. Suppose every $x \in |P|$ has a neighborhood, open in $\mathbb{R}^N$, intersecting only finitely many simplices in P. Then the following hold.*

(i) *P is "locally finite": Each $v \in P^{(0)}$ belongs to only finitely many simplices in P.*
(ii) *$|P|$ is locally compact.*
(iii) *$|P|$ is a subspace of $\mathbb{R}^N$. I.e., the polytope topology of $|P|$ coincides with the topology that $|P|$ inherits from $\mathbb{R}^N$.*

Proof. Suppose $|P| \subset \mathbb{R}^N$ and every $x \in |P|$ has a neighborhood intersecting only finitely many simplices in P. Let $v \in P^{(0)}$. Then v has a neighborhood U that intersects only finitely many simplices in P. If $\sigma \in P$ and $v \in \sigma^{(0)}$, then $v \in \sigma \cap U$. I.e., U intersects σ. Therefore, v is a vertex of only finitely many simplices in P. This proves (i). (See Munkres [Mun84, p. 11].)

By item (i) and Munkres [Mun84, Lemma 2.6, p. 11] we have that $|P|$ is locally compact. And by Munkres [Mun84, Exercise 9, p. 14], the space $|P|$ is a subspace of $\mathbb{R}^N$. □

A simplicial complex P' in $\mathbb{R}^N$ is a "subdivision" of P (Munkres [Mun84, p. 83]) if:

(1) Each simplex in P' is contained in a simplex of P.
(2) Each simplex in P equals the union of finitely many simplices in P'.

In particular, a subdivision of a finite complex is finite.

Proof of lemma B.2. Let $x, y \in |P|$. Since P is a finite complex there exists $\delta_1 > 0$ s.t. if $\rho, \tau \in P$ are disjoint then $dist(\rho, \tau) > 2\delta_1$. Let ρ (τ) be the unique simplex in P s.t. $x \in \operatorname{Int}\rho$ (respectively [resp.], $y \in \operatorname{Int}\tau$; see (B.7)). Therefore, if ρ and τ are disjoint then the Euclidean length $|x - y|$ is bounded below by $2\delta_1$. Moreover, $\left|\boldsymbol{\beta}(z)\right| \leq 1$ for every $z \in |P|$ since the components of $\boldsymbol{\beta}(x)$ are nonnegative and sum to 1. Thus,

$$\left|\boldsymbol{\beta}(x) - \boldsymbol{\beta}(y)\right| \leq (1/\delta_1)|x - y| \text{ if } x \text{ and } y \text{ lie in disjoint simplicies.} \tag{B.9}$$

So assume $\rho \cap \tau \neq \varnothing$. In fact, first consider the behavior of β on a single simplex, ρ in P. (This covers the case where $\tau \subset \rho$ or *vice versa.*) Suppose ρ is an n-simplex, so ρ has $n+1$ vertices $v(0), \ldots, v(n)$. If $n = 0$, i.e., ρ is a single point, then β is trivially Lipschitz on ρ. So suppose $n > 0$. We show that β is Lipschitz on ρ. We can assume $|P| \subset \mathbb{R}^N$ for some $N \geq n$. Let $V^{(n+1)\times N}$ be the matrix whose i^{th} row is $v(i-1)$ ($i = 1, \ldots, n+1$). (Use superscripts to indicate matrix dimension.) Let $V_0^{n\times N}$ be the matrix whose i^{th} row is $v(i) - v(0)$ ($i = 1, \ldots, n$). Let $1_n^{n\times 1}$ be the column vector $(1, \ldots, 1)^T$. Thus,

$$(-1_n \;\; I_n)V = V_0, \tag{B.10}$$

where I_n is the $n \times n$ identity matrix.

The vertices of ρ are geometrically independent so V_0 has full rank n. This means $V_0V_0^T$ is invertible. But by (B.10) $(-1_n \;\; I_n)VV_0^T = V_0V_0^T$. Therefore, $W^{(n+1)\times n} := VV_0^T$ has rank n. This implies that the vector $1_{n+1}^{(n+1)\times 1} = (1, \ldots, 1)^T$ is not in the column space of $W^{(n+1)\times n}$. For suppose for some column vector α we have $W\alpha = 1_{n+1}$. Then $\alpha \neq 0$ and from (B.10) and the fact that $V_0V_0^T$ is nonsingular we have

$$0 \neq V_0V_0^T\alpha = (-1_n \; I_n)W\alpha = (-1_n \; I_n)1_{n+1} = 0.$$

Therefore, $(W, \; 1_{n+1})$ is invertible.

For $x \in \rho$, let $\left(\boldsymbol{\beta}^\rho(x)\right)^{1\times(n+1)}$ be the row vector $\left(\beta_{v(0)}(x), \ldots, \beta_{v(n)}(x)\right)$. Think of $x \in \mathbb{R}^N$ as a row vector. Then we have $x = \boldsymbol{\beta}^\rho(x)V$ and $1 = \boldsymbol{\beta}^\rho(x)1_{n+1}$. Therefore,

$$(xV_0^T, \; 1) = \boldsymbol{\beta}^\rho(x)(W, \; 1_{n+1})^{(n+1)\times(n+1)}.$$

But we have just observed that $U^{(n+1)\times(n+1)} := (W, \; 1_{n+1})$ is invertible. Therefore,

$$\boldsymbol{\beta}^\rho(x) = (xV_0^T, \; 1)U^{-1}.$$

Hence, $\boldsymbol{\beta}^\rho$ is affine on ρ. Therefore, $\boldsymbol{\beta}^\rho$ and, hence, $\boldsymbol{\beta}$ is Lipschitz on ρ. Since P is a finite complex there is $K < \infty$ that works as a Lipschitz constant for every simplex in P. I.e.,

$$\left|\boldsymbol{\beta}(x) - \boldsymbol{\beta}(x')\right| \leq K|x - x'| \text{ for every } x, x' \in \rho \text{ for every } \rho \in P. \tag{B.11}$$

It remains to tackle the case

$$\begin{aligned} &x \in \operatorname{Int}\rho \text{ and } y \in \operatorname{Int}\tau; \;\; \rho, \tau \in P; \\ &\qquad\qquad \rho \cap \tau \neq \varnothing \text{ but } \rho \text{ is not a subset of } \tau \text{ and } \tau \text{ is not a subset of } \rho. \end{aligned} \tag{B.12}$$

$\rho \cap \tau \neq \varnothing$ but ρ is not a subset of τ and τ is not a subset of ρ. In this case, by (B.6), $(\operatorname{Int}\rho) \cap (\operatorname{Int}\tau) = \varnothing$. We handle this case by reducing it to the last case. By (B.5), $\rho \cap \tau$ is a simplex, a proper face of both ρ and τ. Let ξ be the face of ρ opposite $\rho \cap \tau$ and let ω be the face of τ opposite $\rho \cap \tau$. Let $x \in \operatorname{Int}\rho$ and $y \in \operatorname{Int}\tau$.

Claim: There is a unique $z_0 = z_0(x) \in \xi$ s.t. the line passing through x and z_0 intersects $\operatorname{Int}(\rho \cap \tau)$. Given $z \in \xi$, the line, $L(z) = L(z, x)$, passing through z and x is unique since $x \in \operatorname{Int}\rho$ implies $x \notin \xi$. Let $v(0), \ldots, v(n)$ be the vertices of ρ and, renumbering if necessary, we may assume $v(0), \ldots, v(m)$ are the vertices of $\rho \cap \tau$ for some $m = 0, \ldots, n-1$. Then $v(m+1), \ldots, v(n)$ are the vertices of ξ. Let $z \in \xi$ and write

$$z = \sum_{i=m+1}^{n} \mu_i v(i),$$

where the μ_i's are nonnegative and sum to 1.

First, we prove there is at most one $z \in \xi$ s.t. $L(z) \cap \rho \cap \tau \neq \varnothing$. Suppose $L(z)$ intersects $\rho \cap \tau$ at $\tilde{x} = \sum_{i=0}^{m} \mu_i v(i)$. Then for some $t \in \mathbb{R}$ with $t \neq 1$ we have

$$\begin{aligned} \text{(B.13)} \quad \tilde{x} = \sum_{i=0}^{m} \mu_i v(i) &= t \sum_{i=0}^{n} \beta_{v(i)}(x) v(i) + (1-t) \sum_{i=m+1}^{n} \mu_i v(i) \\ &= \sum_{i=0}^{m} t\, \beta_{v(i)}(x) v(i) + \sum_{i=m+1}^{n} \big[t\, \beta_{v(i)}(x) - (t-1)\mu_i \big] v(i). \end{aligned}$$

Then by geometric independence of $v(0), \ldots, v(n)$ we have

$$\text{(B.14)} \qquad \mu_i = t\beta_{v(i)}(x), \quad i = 0, \ldots, m \quad \text{and} \quad \mu_i = \frac{t}{t-1}\beta_{v(i)}(x), \quad i = m+1, \ldots, n.$$

Let $b = \sum_{i=0}^{m} \beta_{v(i)}(x)$. Since $x \in \operatorname{Int}\rho$, we have $b \in (0,1)$. From (B.14) and the fact that $\sum_{i=0}^{m} \mu_i = 1$ we see $t = 1/b > 1$. In particular, z and $\tilde{x}$ are unique if they exist. If it exists, denote that z by z_0.

Next, we prove existence of z_0. Let $t = 1/b$. Then it is easy to see that if $\mu_0, \ldots, \mu_n$ are defined by (B.14) then

$$\sum_{i=0}^{m} \mu_i = 1 = \sum_{i=m+1}^{n} \mu_i.$$

Hence, $z_0 := \sum_{i=m+1}^{n} \mu_i v(i) \in \xi$ and $\tilde{x} := \sum_{i=0}^{m} \mu_i v(i) \in \rho \cap \tau$ and (B.13) holds. Since $x \in \operatorname{Int}\rho$, we have $\beta_{v(i)}(x) > 0$ for $i = 1, \ldots, n$. Therefore, $\mu_i > 0$ for $i = 1, \ldots, m$. Thus, $\tilde{x} \in \operatorname{Int}(\rho \cap \tau)$. I.e., $z_0 \in \xi$, x, and $\tilde{x} \in \operatorname{Int}(\rho \cap \tau)$ lie on the same line. This proves the claim. Define $\tilde{y} \in \rho \cap \tau$ similarly. It has similar properties.

The idea behind the rest of the proof is to first show that

$$\text{(B.15)} \qquad |x - \tilde{x}| + |\tilde{x} - \tilde{y}| + |\tilde{y} - y| \leq K' \, |x - y|,$$

where $K' = K'(\rho, \tau) < \infty$ depends only on ρ and τ, not on x or y. Notice that x and $\tilde{x}$ lie in the same simplex in P, *viz.* ρ. Similarly, $\tilde{x}$ and $\tilde{y}$ both lie in $\rho \cap \tau \in P$. The points $\tilde{y}$ and y also lie in the same simplex in P. So we may apply (B.11) to each term in $|x - \tilde{x}| + |\tilde{x} - \tilde{y}| + |\tilde{y} - y|$ and then maximize $K'(\rho, \tau)$ over appropriate pairs $\rho, \tau \in P$.

The simplex $\rho \cap \tau$ lies on a unique plane, $\Pi_{\rho\cap\tau}$, of minimum dimension. (See (B.1).) ($\Pi_{\rho\cap\tau}$ might not pass through the origin.) So, e.g., if $\rho \cap \tau$ is a single point v (i.e., $\rho \cap \tau$ 0-dimensional)

then $\Pi_{\rho\cap\tau} = \{v\}$. Now, $x \in \operatorname{Int}\rho$ so $x \notin \Pi_{\rho\cap\tau}$. Let $\hat{x} \in \Pi_{\rho\cap\tau}$ be the orthogonal projection of x onto $\Pi_{\rho\cap\tau}$, i.e., $\hat{x}$ is the closest point of $\Pi_{\rho\cap\tau}$ to x. Note that $\hat{x}$ may not lie in $\rho \cap \tau$. Define $\hat{y}$ similarly. Let x_0 be an arbitrary point in $\operatorname{Int}(\rho\cap\tau)$. E.g., x_0 might be the barycenter of $\rho\cap\tau$. (See (B.3).) In any case, x_0 need only depend on $\rho\cap\tau$, not on x or y.Let

$$y_0 := x_0. \tag{B.16}$$

Then by (B.6), there exists $r > 0$ s.t. the distance from $x_0 = y_0$ to any face of ρ or τ that does not itself have $\rho\cap\tau$ as a face is at least $2r$. We may assume r only depends on $\rho\cap\tau$, not on x or y.

Claim:

$$\dot{x} := x_0 + |x - \hat{x}|^{-1} r(x - \hat{x}) \in \rho \text{ and } \dot{y} := y_0 + |y - \hat{y}|^{-1} r(y - \hat{y}) \in \tau. \tag{B.17}$$

First, note that

$$\text{for } t > 0 \text{ sufficiently small, } x_0 + t(x - \hat{x}) \in \operatorname{Int}\rho. \tag{B.18}$$

To see this, observe that by (B.1) we can write

$$\hat{x} = \sum_{i=0}^{m} \zeta_i v(i),$$

where $v(0), \ldots, v(m)$ are the vertices of $\rho\cap\tau$; $\zeta_0, \ldots, \zeta_m \in \mathbb{R}$; and $\zeta_0 + \cdots + \zeta_m = 1$. (But the ζ_i's do not have to be nonnegative.) Moreover, since x_0 is an interior point of $\rho\cap\tau$ we have

$$\beta_{v(i)}(x_0) > 0, \text{ for } i = 0, \ldots, m, \text{ but } \beta_{v(i)}(x_0) = 0 \text{ for } i = m+1, \ldots, n.$$

Let $t > 0$. Then

$$x_0 + t(x - \hat{x}) = \sum_{i=0}^{m} \big(\beta_{v(i)}(x_0) - t\zeta_i + t\beta_{v(i)}(x)\big) v(i) + t \sum_{i=m+1}^{n} \beta_{v(i)}(x) v(i). \tag{B.19}$$

Since $\beta_{v(i)}(x_0) > 0$ for $i = 0, \ldots, m$, for $t > 0$ sufficiently small $\beta_{v(i)}(x_0) - t\zeta_i > 0$ for $i = 0, \ldots, m$. So certainly $\beta_{v(i)}(x_0) - t\zeta_i + t\beta_{v(i)}(x) > 0$ for $i = 0, \ldots, m$. I.e., the coefficients in (B.19) are all strictly positive. Finally, the sum of the coefficients satisfies

$$\begin{aligned}\sum_{i=0}^{m} \big(\beta_{v(i)}(x_0) - t\zeta_i + t\beta_{v(i)}(x)\big) + t \sum_{i=m+1}^{n} \beta_{v(i)}(x) &= \sum_{i=0}^{m} \beta_{v(i)}(x_0) - t \sum_{i=0}^{m} \zeta_i + t \sum_{i=0}^{n} \beta_{v(i)}(x) \\ &= 1 - t + t \\ &= 1.\end{aligned}$$

That completes the proof of (B.18).

Now suppose $\dot{x}$ defined by (B.17) does *not* lie in ρ. Let Π_ρ be the smallest plane in $\mathbb{R}^N$ containing ρ. So $\Pi_{\rho\cap\tau} \subset \Pi_\rho$. By (B.1), we have

$$\Pi_\rho = \left\{ \sum_{i=0}^{n} \gamma_i v(i) : \sum_{i=0}^{n} \gamma_i = 1 \right\} = \left\{ v(0) + \sum_{i=1}^{n} \gamma_i \big(v(i) - v(0)\big) : \gamma_1, \ldots, \gamma_n \in \mathbb{R} \right\},$$

where $v(0), \ldots, v(n)$ are the vertices of ρ. Since $v(1) - v(0), \ldots, v(n) - v(0)$ are linearly independent, the map that takes a point $\sum_{i=0}^{n} \gamma_i v(i) \in \Pi_\rho$ to the vector $\gamma_0, \ldots, \gamma_n$ is well-defined and continuous. Now $x_0 \in \rho\cap\tau \subset \Pi_\rho$, $x \in \rho \subset \Pi_\rho$, and $\hat{x} \in \Pi_{\rho\cap\tau} \subset \Pi_\rho$. Moreover, the

coefficients of x_0, x, and $\hat{x}$ in the expression for $\dot{x}$ in (B.17), *viz.*, 1, $r/|x-\hat{x}|$, and $-r/|x-\hat{x}|$ sum to 1. It follows that $\dot{x} \in \Pi_\rho$. Hence, we can write $\dot{x} = \sum_{i=0}^n \zeta_i v(i)$ with $\zeta_0 + \cdots + \zeta_n = 1$.

Let S be the line segment joining x_0 and $\dot{x}$. I.e.,

$$S = \big\{x_0 + t(x-\hat{x}) : 0 \leq t \leq r/|x-\hat{x}|\big\}. \tag{B.20}$$

By (B.18) for some $t \in (0, r/|x-\hat{x}|)$ we have

$$x' := x_0 + t(x-\hat{x}) \in (\operatorname{Int}\rho) \cap S. \tag{B.21}$$

Since $x' \in \operatorname{Int}\rho$, the coefficients in the representation of x' as a linear combination of $v(0), \ldots, v(n)$ must all be strictly positive. Since by assumption $\dot{x} \notin \rho$, one or more of the coefficients, $\zeta_0, \ldots, \zeta_n$, of $v(0), \ldots, v(n)$ for $\dot{x}$ must be strictly negative. Therefore, somewhere between x' and $\dot{x}$ the segment S must cross the boundary $\operatorname{Bd}\rho$. Let $w \in \operatorname{Bd}\rho$ be the point of intersection. Thus, for some $s \in (t, r/|x-\hat{x}|)$ we have

$$w = x_0 + s(x-\hat{x}). \tag{B.22}$$

Let ω be the, necessarily proper, face of ρ s.t. $w \in \operatorname{Int}\omega$. (See (B.7).) Now, $\rho\cap\tau$ cannot be a face of ω. For suppose $\rho\cap\tau \subset \omega$. Note that $w \neq x_0$, because otherwise $s(x-\hat{x}) = 0$ in (B.22), an impossibility since $x \neq \hat{x}$ and $s > 0$. Hence, under the assumption that $\rho\cap\tau\subset\omega$ the segment S contains two distinct points of ω, *viz.*, $x_0 \in \rho\cap\tau$ and w. As a proper face of ρ, the simplex ω is defined by the vanishing of some set of barycentric coordinates. Thus, there exists a nonempty proper subset J of $\{0, \ldots, n\}$ s.t.

$$\omega = \left\{ \sum_{j=0}^n \beta_j v(j) : \beta_j \geq 0 \ (j = 0, \ldots, n), \beta_j = 0 \text{ if } j \in J, \text{ and } \sum_{j=0}^n \beta_j = 1 \right\}.$$

Since $x, \hat{x} \in \Pi_\rho$, for some $\gamma_0, \ldots, \gamma_n \in \mathbb{R}$ we have

$$x - \hat{x} = \sum_{j=0}^n \gamma_j v(j), \text{ where } \sum_{j=0}^n \gamma_j = 0.$$

Under the hypothesis that $\rho\cap\tau\subset\omega$, we have $w, x_0 \in \omega$. In particular, we have $\beta_{v(j)}(x_0) = 0$ for $j \in J$. It follows from (B.22) that $\gamma_j = 0$ if $j \in J$. Hence, by (B.20) for every $x'' \in S \subset \Pi_\rho$ we can write (uniquely)

$$x'' = \sum_{j \in J^c} \alpha_j v(j), \text{ where } \sum_{j\in J^c} \alpha_j = 1.$$

(Here, $J^c = \{j = 0, \ldots, n : j \notin J\}$.) In particular, $S \cap (\operatorname{Int}\rho) = \varnothing$. But by (B.21), $x' \in S\cap(\operatorname{Int}\rho)$. Contradiction. This proves $\rho\cap\tau$ cannot be a face of ω.

Since $\rho\cap\tau$ is not a face of ω, by choice of $r > 0$ the distance from x_0 to ω is at least $2r$. Since ω lies between x_0 and $\dot{x}$ along S we have by (B.17)

$$r = |\dot{x} - x_0| \geq 2r > 0.$$

This contradiction proves the claim (B.17).

Claim: The angle between $x-\hat{x}$ and $y-\hat{y}$ is bounded away from 0. I.e., there exists $\gamma \in (0,1)$ independent of $x \in \operatorname{Int}\rho$ and $y \in \operatorname{Int}\tau$ (i.e., γ only depends on ρ and τ) s.t.

$$(x-\hat{x})\cdot(y-\hat{y}) \leq \gamma|x-\hat{x}||y-\hat{y}|, \tag{B.23}$$

where, as usual, "$\cdot$" indicates the usual Euclidean inner product. Suppose (B.23) is false. Then there exist sequences $\{x_n\} \subset \operatorname{Int}\rho$, $\{y_n\} \subset \operatorname{Int}\tau$ s.t.

$$\frac{(x_n - \hat{x}_n) \cdot (y_n - \hat{y}_n)}{|x_n - \hat{x}_n||y_n - \hat{y}_n|} \to 1,$$

where $\hat{x}_n$ ($\hat{y}_n$) is the orthogonal projection of x_n (resp. y_n) onto $\Pi_{\rho\cap\tau}$. Define $\dot{x}_n$ as in (B.17) with x and $\hat{x}$ replaced by x_n and $\hat{x}_n$, resp. Define $\dot{y}_n$ similarly. By definition of $\dot{x}_n$ and $\hat{x}_n$ the vector $\dot{x}_n - x_0$ has length $r > 0$ and is orthogonal to $\Pi_{\rho\cap\tau}$. Ditto for $\dot{y}_n - y_0$. But $x_0 \in \rho \cap \tau \subset \Pi_{\rho\cap\tau}$. Hence, $dist(\dot{x}_n, \rho\cap\tau) \geq r$. Moreover, by (B.17), $\dot{x}_n \in \rho$. Similarly, $dist(\dot{y}_n, \rho\cap\tau) \geq r$ and $\dot{y}_n \in \tau$. Therefore, by compactness of ρ and τ, we may assume $\dot{x}_n \to \dot{x}_\infty \in \rho$ and $\dot{y}_n \to \dot{y}_\infty \in \tau$. We must have $|\dot{x}_\infty - x_0| = r$, $|\dot{y}_\infty - y_0| = r$, $dist(\dot{x}_\infty, \rho\cap\tau) \geq r$, and $dist(\dot{y}_\infty, \rho\cap\tau) \geq r$. In particular,

$$\dot{x}_\infty \in \rho \setminus (\rho\cap\tau) \text{ and } \dot{y}_\infty \in \tau \setminus (\rho\cap\tau) \tag{B.24}$$

Now, by definition of $\{x_n\}$, $\{y_n\}$, $\{\dot{x}_n\}$, and $\{\dot{y}_n\}$, we have

$$(\dot{x}_n - x_0)\cdot(\dot{y}_n - y_0) = r^2 \frac{(x_n - \hat{x}_n) \cdot (y_n - \hat{y}_n)}{|x_n - \hat{x}_n||y_n - \hat{y}_n|} \to r^2 = |\dot{x}_\infty - x_0||\dot{y}_\infty - y_0| \text{ as } n \to \infty.$$

But,

$$(\dot{x}_n - x_0)\cdot(\dot{y}_n - y_0) \to (\dot{x}_\infty - x_0)\cdot(\dot{y}_\infty - y_0) \text{ as } n \to \infty.$$

This means $\dot{x}_\infty - x_0$ and $\dot{y}_\infty - y_0$ are positive multiples of each other. But $\dot{x}_\infty - x_0$ and $\dot{y}_\infty - y_0$ have the same length r. Hence, $\dot{x}_\infty - x_0 = \dot{y}_\infty - y_0$. However, by (B.16), $y_0 = x_0$. Therefore, $\dot{x}_\infty = \dot{y}_\infty$. In particular, $\dot{x}_\infty, \dot{y}_\infty \in \rho\cap\tau$. This contradicts (B.24). The claim (B.23) follows.

By definition of $\hat{x}$ and $\hat{y}$ and (B.23), we have

$$\begin{aligned}
|x-y|^2 &= \big|(x-\hat{x}) + (\hat{x}-\hat{y}) + (\hat{y}-y)\big|^2 \\
&= |x-\hat{x}|^2 + |\hat{x}-\hat{y}|^2 - 2(x-\hat{x})\cdot(y-\hat{y}) + |\hat{y}-y|^2 \\
&\geq |x-\hat{x}|^2 + |\hat{x}-\hat{y}|^2 - 2\gamma|x-\hat{x}||y-\hat{y}| + |y-\hat{y}|^2 \\
&= (1-\gamma)\big(|x-\hat{x}|^2 + |y-\hat{y}|^2\big) + |\hat{x}-\hat{y}|^2 + \gamma\big(|x-\hat{x}| - |y-\hat{y}|\big)^2 \\
&\geq (1-\gamma)\big(|x-\hat{x}|^2 + |y-\hat{y}|^2 + |\hat{x}-\hat{y}|^2\big).
\end{aligned} \tag{B.25}$$

Recall the inequality

$$2a^2 + 2b^2 \geq (a+b)^2, \quad (a, b \in \mathbb{R}). \tag{B.26}$$

Applying this twice to (B.25) we get

$$\begin{aligned}
|x-y|^2 &\geq \frac{1-\gamma}{4}\Big(2\big[|x-\hat{x}| + |y-\hat{y}|\big]^2 + 4|\hat{x}-\hat{y}|^2\Big) \\
&\geq \frac{1-\gamma}{4}\Big(2\big[|x-\hat{x}| + |y-\hat{y}|\big]^2 + 2|\hat{x}-\hat{y}|^2\Big) \\
&\geq \frac{1-\gamma}{4}\big[|x-\hat{x}| + |y-\hat{y}| + |\hat{x}-\hat{y}|\big]^2.
\end{aligned}$$

We conclude

$$\frac{2}{\sqrt{1-\gamma}}|x-y| \geq |x-\hat{x}| + |\hat{x}-\hat{y}| + |y-\hat{y}|, \text{ for } x \in \operatorname{Int}\rho,\ y \in \operatorname{Int}\tau. \tag{B.27}$$

Claim: The angle, θ, between $x - \tilde{x}$ and $\Pi_{\rho\cap\tau}$ is bounded away from 0. Since $\hat{x}$ is the orthogonal projection of x onto $\Pi_{\rho\cap\tau}$, we have that θ is the angle between $x-\tilde{x}$ and $\hat{x}-\tilde{x}$ and $\sin\theta = |x-\hat{x}|/|x-\tilde{x}|$. By definition of $\hat{x}$, $|x-\tilde{x}|/|x-\hat{x}| \geq 1$. Therefore, θ being bounded away from 0 is equivalent to

(B.28) $1/\sin\theta = |x-\tilde{x}|/|x-\hat{x}|$ is bounded above by some $\alpha \in (1,\infty)$ independent of $x \in \operatorname{Int}\rho$.

And similarly for y, $\tilde{y}$, and $\hat{y}$.

If $z \in \xi$ (the face of ρ opposite $\rho\cap\tau$), let $\hat{z}$ denote the orthogonal projection of z onto $\Pi_{\rho\cap\tau}$. Recall that $\tilde{x}$, x, and z_0 lie on the same line. Taking orthogonal projections, we see that $\tilde{x}$, $\hat{x}$, and $\hat{z}_0$ lie on the same line in $\Pi_{\rho\cap\tau}$. Therefore, by similarity of triangles[2],

$$\frac{|x-\tilde{x}|}{|x-\hat{x}|} = \frac{|z_0-\tilde{x}|}{|z_0-\hat{z}_0|}.$$

But since ξ and $\rho\cap\tau$ are disjoint and compact, $|z-\hat{z}|$ is bounded below and $|z-w|$ is bounded above in $(z,w) \in \xi \times (\rho\cap\tau)$. The claim (B.28) follows. Of course, the same thing goes for y and we may assume the same α works for both ρ and τ.

It follows from (B.28) and the Pythagorean theorem that

$$|\tilde{x}-\hat{x}| \leq \sqrt{\alpha^2-1}\,|x-\hat{x}| < \alpha|x-\hat{x}|. \text{ Similarly for } y, \tilde{y}, \text{ and } \hat{y}. \tag{B.31}$$

Consequently,

$$\begin{aligned}|\tilde{x}-\tilde{y}| \leq |\tilde{x}-\hat{x}| + |\hat{x}-\hat{y}| + |\hat{y}-\tilde{y}| &\leq \alpha|x-\hat{x}| + |\hat{x}-\hat{y}| + \alpha|y-\hat{y}| \\ &\leq \alpha|x-\hat{x}| + 2\alpha|\hat{x}-\hat{y}| + \alpha|y-\hat{y}|,\end{aligned}$$

since $\alpha > 1$. Hence,

$$|\hat{x}-\hat{y}| \geq \frac{1}{2\alpha}|\tilde{x}-\tilde{y}| - \frac{1}{2}|x-\hat{x}| - \frac{1}{2}|y-\hat{y}|.$$

Substituting this into (B.27) we get

$$\frac{2}{\sqrt{1-\gamma}}|x-y| \geq \frac{1}{2}|x-\hat{x}| + \frac{1}{2\alpha}|\tilde{x}-\tilde{y}| + \frac{1}{2}|y-\hat{y}|.$$

Therefore, by (B.28) again,

$$\frac{2}{\sqrt{1-\gamma}}|x-y| \geq \frac{1}{2\alpha}\big(|x-\tilde{x}| + |\tilde{x}-\tilde{y}| + |y-\tilde{y}|\big). \tag{B.32}$$

[2]To see all this analytically, let $c = |x-\tilde{x}|/|z_0-\tilde{x}|$. ($|z_0-\tilde{x}| > 0$, since $\tilde{x} \in \rho\cap\tau$ and $z_0 \in \xi$, the face opposite $\rho\cap\tau$.) Then

$$x = c(z_0-\tilde{x}) + \tilde{x}, \tag{B.29}$$

since x lies on the line segment joining z_0 and $\tilde{x}$. Let $\ddot{x} = c(\hat{z}_0-\tilde{x}) + \tilde{x}$. Then $\ddot{x}$ lies on the line joining $\tilde{x}$ and $\hat{z}_0$. (In particular, $\ddot{x} \in \Pi_{\rho\cap\tau}$.) But it is easy to see from (B.29) that $x - \ddot{x} = c(z_0-\hat{z}_0) \perp \Pi_{\rho\cap\tau}$. I.e.,

$$\hat{x} = \ddot{x} = c(\hat{z}_0-\tilde{x}) + \tilde{x}. \tag{B.30}$$

Thus, $z_0-\tilde{x}$, $x-\tilde{x}$, $\hat{x}-\tilde{x}$, and $\hat{z}_0-\tilde{x}$ lie in the subspace spanned by $z_0-\tilde{x}$ and $\hat{z}_0-\tilde{x}$ and

$$\frac{|x-\tilde{x}|}{|x-\hat{x}|} = \frac{\Big|\big[c(z_0-\tilde{x})+\tilde{x}\big]-\tilde{x}\Big|}{\Big|\big[c(z_0-\tilde{x})+\tilde{x}\big]-\big[c(\hat{z}_0-\tilde{x})+\tilde{x}\big]\Big|} = \frac{|z_0-\tilde{x}|}{|z_0-\hat{z}_0|}$$

by (B.29) and (B.30).

I.e., if (B.12) holds

$$\frac{4\alpha}{\sqrt{1-\gamma}}|x-y| \geq |x-\tilde{x}| + |\tilde{x}-\tilde{y}| + |y-\tilde{y}|. \tag{B.33}$$

Let $K' = K'(\rho, \tau) := \frac{4\alpha}{\sqrt{1-\gamma}}$. Then (B.15) holds. (Maximizing over all appropriate $\rho, \tau \in P$ yields corollary B.3.) (B.33) and (B.11) together imply

$$\begin{aligned} \big|\boldsymbol{\beta}(x) - \boldsymbol{\beta}(y)\big| &\leq \big|\boldsymbol{\beta}(x) - \boldsymbol{\beta}(\tilde{x})\big| + \big|\boldsymbol{\beta}(\tilde{x}) - \boldsymbol{\beta}(\tilde{y})\big| + \big|\boldsymbol{\beta}(\tilde{y}) - \boldsymbol{\beta}(y)\big| \\ &\leq K\big(|x-\tilde{x}| + |\tilde{x}-\tilde{y}| + |y-\tilde{y}|\big) \\ &\leq KK'(\rho,\tau)|x-y|. \end{aligned}$$

Now maximize over all $\rho, \tau \in P$. This completes the proof. □

C. Lipschitz maps and Hausdorff measure and dimension

Hausdorff dimension (Giaquinta *et al* [GMS98, p. 14, Volume I] and Falconer [Fal90, p. 28]) is defined as follows. First, we define Hausdorff measure (Giaquinta *et al* [GMS98, p. 13, Volume I], Hardt and Simon [HS86, p. 9], and Federer [Fed69, 2.10.2. p. 171]). Let $s \geq 0$. If s is an integer, let ω_s denote the volume of the unit ball in $\mathbb{R}^s$:

$$\omega_s = \frac{\Gamma(1/2)^s}{\Gamma\left(\frac{s}{2}+1\right)}, \tag{C.1}$$

where Γ is Euler's gamma function (Federer [Fed69, pp. 135, 251]). If s is not an integer, then ω_s can be any convenient positive constant. Federer uses (C.1) for any $s \geq 0$. Let X be a metric space with metric d_X. For any subset A of X and $\delta > 0$ first define

$$\mathcal{H}^s_\delta(A) = \omega_s \inf \left\{ \sum_j \left(\frac{diam(C_j)}{2} \right)^s \right\}. \tag{C.2}$$

Here, "*diam*" is diameter (w.r.t. d_X) and the infimum is taken over all (at most) countable collections $\{C_j\}$ of subsets of X with $A \subset \bigcup_j C_j$ and $diam\, C_j < \delta$. (Thus, $\mathcal{H}^s_\delta(\varnothing) = 0$ since an empty cover covers $\varnothing$ and an empty sum is 0. If X is second countable, it follows from Lindelöf's theorem, Simmons [Sim63, Theorem A, p. 100], that for any $\delta > 0$, such a countable cover exists. Otherwise, $\mathcal{H}^s_\delta(A) = +\infty$.) We may assume that the covering sets C_j are all open or that they are closed (Federer [Fed69, 2.10.2, p. 171]). The s-dimensional Hausdorff measure of A is then

$$\mathcal{H}^s(A) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A). \tag{C.3}$$

Note that $\mathcal{H}^0(A)$ is the cardinality of A if it is finite. Otherwise, $\mathcal{H}^0(A) = +\infty$. At the other extreme,

$$A = \varnothing \text{ if and only if } \mathcal{H}^0(A) = 0. \tag{C.4}$$

For every $s \geq 0$, $\mathcal{H}^s$ is an outer measure on X and the Borel subsets of X are $\mathcal{H}^s$-measurable (Federer [Fed69, 2.10.2, p. 171] and Hardt and Simon [HS86, p. 10]). Note that if X is a subset of a Euclidean space (and inherits the Euclidean metric) and we rescale X by multiplying each vector in X by $\lambda > 0$, then for every $A \subset X$ the measure $\mathcal{H}^s(A)$ will be replaced by $\lambda^s \mathcal{H}^s(A)$.

For $A \subset X$ nonempty there will be a number $s_0 \in [0, +\infty]$ s.t. $0 \leq s < s_0$ implies $\mathcal{H}^s(A) = +\infty$ and $s > s_0$ implies $\mathcal{H}^s(A) = 0$. That number s_0 is the "Hausdorff dimension", $\dim A$, of A (Falconer [Fal90, p. 28]). (In particular, $\dim \varnothing = 0$. In appendix B we already defined $\dim \sigma$, where σ is a simplex, and $\dim P$, where P is a simplicial complex. These dimensions are the same as the respective Hausdorff dimensions, at least if P is finite.) But $\mathcal{H}^s(A) = 0$ is a stronger statement than $\dim A \leq s$ (Falconer [Fal90, p. 29]). It is easy to see that

(C.5) if A is a finite union of Borel measurable sets $A_1, \ldots, A_k$
then $\dim A = \max\{\dim A_1, \ldots, \dim A_k\}$

(Falconer [Fal90, p. 29]).

Let Y be a metric space with metric d_Y and let $f : X \to Y$. Recall that f is "Lipschitz(ian)" (Giaquinta *et al* [GMS98, p. 202, Volume I], Falconer [Fal90, p. 8], Federer [Fed69, pp. 63 – 64]) if there exists $K < \infty$ (called a "Lipschitz constant" for f) s.t.

$$d_Y\big[f(x), f(y)\big] \leq K\, d_X(x, y), \qquad \text{for every } x, y \in X.$$

Example C.1. If $\mathcal{S} \subset X$ is compact then the function $y \mapsto dist(y, \mathcal{S}) \in \mathbb{R}$ is Lipschitz with Lipschitz constant 1.

Further recall the following. Let $k = 1, 2, \ldots$ and let $\mathcal{L}^k$ denote k-dimensional Lebesgue measure. Suppose T is a linear operator on $\mathbb{R}^k$ and $v \in \mathbb{R}^k$. Then by Rudin [Rud66, Theorems 8.26(a) and 8.28, pp. 173–174] if $A \subset \mathbb{R}^k$ is Borel measurable then $T(A) + v$ is Lebesgue measurable and

$$\mathcal{L}^k\big[T(A) + v\big] = |\det T|\, \mathcal{L}^k(A). \tag{C.6}$$

This motivates the following basic fact about Hausdorff measure and dimension (Falconer [Fal90, p. 28], Hardt and Simon [HS86, 1.3, p. 11]). Let $f : X \to Y$ be Lipschitz with Lipschitz constant K. Then for $s \geq 0$,

$$\mathcal{H}^s\big[f(X)\big] \leq K^s\, \mathcal{H}^s(X). \text{ Therefore, } \dim f(X) \leq \dim X. \tag{C.7}$$

$f : X \to Y$ is "locally Lipschitz" (Federer [Fed69, pp. 64]) if each $x \in X$ has a neighborhood, V, s.t. the restriction $f|_V$ is Lipschitz. So any Lipschitz map is locally Lipschitz and, conversely, any locally Lipschitz function on X is Lipschitz on any compact subset of X. Moreover,

(C.8) The composition of (locally) Lipschitz maps is (resp., locally) Lipschitz.

An easy consequence of (C.7) is the following.

Lemma C.2. *Let $X \subset \mathbb{R}^k$. Suppose $f : X \to Y$ is locally Lipschitz. If $s \geq 0$ and $\mathcal{H}^s(X) = 0$, then $\mathcal{H}^s\big[f(X)\big] = 0$. In particular, $\dim f(X) \leq \dim X$. We also have $\mathcal{H}^0\big[f(X)\big] \leq \mathcal{H}^0(X)$.*

Proof. By Lindelöf's theorem (Simmons [Sim63, Theorem A, p. 100]) X can be partitioned into a countable number of disjoint Borel sets $A_1, A_2, \ldots$ on each of which f is Lipschitz with respective Lipschitz constant K_i. By (C.7), we have

$$\mathcal{H}^s\big[f(X)\big] \leq \sum_i \mathcal{H}^s\big[f(A_i)\big] \leq \sum_i K_i^s \mathcal{H}^s(A_i).$$

□

Another generalization of (C.7) is the following.

Lemma C.3. *Let k and m be positive integers. Let $U \subset \mathbb{R}^k$ be open and suppose $h = (h_1, \ldots, h_m) : U \to \mathbb{R}^m$ is continuously differentiable. For $x = (x_1, \ldots, x_k) \in U$, let $Dh(x)$ be the $m \times k$ Jacobian matrix*

$$Dh(x) = \left(\frac{\partial h_i(y)}{\partial y_j} \right)_{y=x}.$$

At each $x \in U$, let $\lambda(x)^2$ be the largest eigenvalue of $Dh(x)^T Dh(x)$ (with $\lambda(x) \geq 0$; "T" indicates matrix transposition.) Then λ is continuous. Furthermore, let $a \geq 0$ and let $A \subset U$ be Borel with $\mathcal{H}^a(A) < \infty$. Then

$$\mathcal{H}^a\big[h(A)\big] \leq \int_A \lambda(x)^a \, \mathcal{H}^a(dx). \tag{C.9}$$

Proof. By lemma A.6 and and continuity of Dh, λ is continuous. Let $\epsilon > 0$. Since λ is continuous, by Lindelöf's theorem (Simmons [Sim63, Theorem A, p. 100]), there exists an at most countable cover, $C_1, C_2, \ldots$, of U by open convex sets with the property

$$x, x' \in C_i \Rightarrow \big|\lambda(x)^a - \lambda(x')^a\big| < \epsilon, \quad (i = 1, 2, \ldots). \tag{C.10}$$

For each $i = 1, 2, \ldots$ let $\Lambda_i = \sup_{x \in C_i} \lambda(x)$. We prove the *claim:* on each C_i, the function h is Lipschitz with Lipschitz constant Λ_i. (See Giaquinta *et al* [GMS98, Theorem 2, p. 202, Vol. I].) Let $x, y \in C_i$. Think of x, y as row vectors. Since C_i is open and convex there is an open interval $I \supset [0,1]$ s.t. for every $u \in I$ we have $(1-u)x + uy \in C_i$. The function $f : u \mapsto h\big[(1-u)x + uy\big] \in \mathbb{R}^m$ $(u \in I)$ is defined and differentiable. It defines an arc in $\mathbb{R}^m$. By the area formula (Hardt and Simon [HS86, p. 13])

$$\big|h(y) - h(x)\big| \leq \text{length of arc } f = \int_0^1 \big|f'(u)\big| \, du = \int_0^1 \Big|Dh[(1-u)x + uy](y-x)^T\Big| \, du. \tag{C.11}$$

Let $u \in [0,1]$ and let $w = (1-u)x + uy \in C_i \subset U \subset \mathbb{R}^k$. Let $\lambda_1^2 \geq \lambda_2^2 \geq \cdots \geq \lambda_k^2 \geq 0$ be the eigenvalues of $Dh(w)^T Dh(w)$, so $\lambda^2(w) = \lambda_1^2$. Let $z_1, \ldots, z_k \in \mathbb{R}^k$ be corresponding orthonormal eigenvectors, thought of as row vectors. Write $y - x = \sum_{j=1}^k \alpha_j z_j$. Then

$$\begin{aligned}
\big|Dh(w)(y-x)^T\big|^2 &= (y-x)\big(Dh(w)^T Dh(w)\big)(y-x)^T \\
&= \left(\sum_{j=1}^k \alpha_j z_j\right) \big(Dh(w)^T Dh(w)\big) \left(\sum_{j=1}^k \alpha_j z_j^T\right) \\
&= \left(\sum_{j=1}^k \alpha_j z_j\right) \left(\sum_{j=1}^k \alpha_j \lambda_j^2 z_j^T\right) \\
&= \sum_{j=1}^k \lambda_j^2 \alpha_j^2 \\
&\leq \lambda^2(w) \sum_{j=1}^k \alpha_j^2 \\
&\leq \Lambda_i^2 |y-x|^2.
\end{aligned}$$

I.e.,

$$\left|Dh(w)(y-x)^T\right| \leq \Lambda_i |y-x|.$$

Substituting this into (C.11) proves the claim.

Let $A_1 = A \cap C_1$. Having defined $A_1, \ldots, A_n$, let

$$A_{n+1} = (A \cap C_{n+1}) \setminus \left(\bigcup_{i=1}^{n} A_i \right).$$

Then $A_1, A_2, \ldots$ is a Borel partition of A. By (C.7) and (C.10),

$$\mathcal{H}^a\big[h(A)\big] \leq \sum_i \mathcal{H}^a\big[h(A_i)\big] \leq \sum_i \Lambda_i^a \mathcal{H}^a(A_i) \leq \int_A \lambda(x)^a \mathcal{H}^a(dx) + \mathcal{H}(A)\epsilon.$$

Since $\epsilon > 0$ is arbitrary and $\mathcal{H}^a(A) < \infty$, the lemma follows. □

The following fact is useful. For the proof see the preceding proof of lemma C.3.

Corollary C.4. *Let $k, m > 0$ be integers. Let $U \subset \mathbb{R}^k$ be open and let $h : U \to \mathbb{R}^m$ be continuously differentiable. Then h is locally Lipschitz. In particular, if $A \subset U$ is compact then h is Lipschitz on A.*

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Unit 42, NYSPI, 1051 Riverside Dr., New York, NY 10032, U.S.A.
E-mail address: spe4@columbia.edu